Vortex loops in $\mathcal{N}=2$ supersymmetric gauge theories on three-manifolds

($\mathbf{3}$次元多様体上の $\boldsymbol{\mathcal{N}=2}$超対称ゲージ理論の渦ループ)

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Abstract

In this thesis we study the two different descriptions of BPS vortex loops in 3D $\mathcal{N}=2$ supersymmetric (SUSY) non-abelian gauge theories and present their equivalence. First, we calculate the expectation value of BPS vortex loops on an ellipsoid using a definition that involves performing a path integral over the field with a prescribed singular behavior. By using the obtained result, we revisit the known equivalence between Wilson and vortex loops in pure Chern-Simons theory. This implies an alternative definition of BPS vortex loops, where a quantum mechanics on a loop interacts with the 3D field theory. However, straightforward computations of expectation values in the $\mathcal{N}=2$ SUSY theory lead to an undesired shift in the correspondence rule for parameters. To address this issue, we propose a relation between the parameter shift and the global anomaly of $\mathcal{N}=2$ SUSY quantum mechanics. Additionally, for theories with $U(N)$ gauge group, we also develop an alternative description of vortex loops in terms of 1D $\mathcal{N}=2$ SUSY gauged linear sigma models (GLSMs) on their worldline. Our construction reproduces certain GLSMs for vortex loops in $\mathcal{N}=4$ theories studied by Assel and Gomis.

Contents

Introduction

Quantum field theory is the most standard way to formulate particle physics and is currently being studied in various direction. One of the subjects that has attracted the attention of many physicists and mathematicians is the theory with supersymmetry(SUSY), which is symmetry with respect to the exchange of bosons and fermions. SUSY provides powerful tools to analyze problems in quantum field theories. A significant progress is the Localization techniques that allows us to compute SUSY-preserving observables exactly.

The localization technique was applied to 3D SUSY gauge theories on S^3 in [1], where a formula for partition function and Wilson loop [2] was obtained for a class of $\mathcal{N} \geq 2$ superconformal Chern-Simons(CS) matter theories. The papers [3,4] generalized the result to the $\mathcal{N} = 2$ supersymmetric theories, in which R-charge of the matter fields is no longer constrained by their Weyl weight. Further generalization was found by [5] that studied theories on the so-called 3D ellipsoid or squashed S^3.

The essential idea of localization is that contributions of a supersymmetric path integral are only from the configuration of bosonic fields known as saddle points. The infinite-dimensional path integral is then reduced to a finite-dimensional integral over the saddle points. The first purpose of this thesis is to review the localization techniques on three-dimensional manifolds and derive the exact formula for the partition function via a supersymmetric path integral. Our main interest is $\mathcal{N} = 2$ SUSY gauge theories on 3D ellipsoid, for which the formula is given by

$$Z_{S^3_b} = \frac{1}{|\mathcal{W}|} \int \mathrm{d}^r \hat{\sigma} e^{-S} \cdot \Delta^{\mathrm{v}}_{\text{1-loop}} \cdot \Delta^{\mathrm{c}}_{\text{1-loop}}.$$

This is a finite-dimensional integral over the saddle point parameter $\hat{\sigma}$ which takes values in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}(G)$. The integrand, a function of $\hat{\sigma}$ and squashing parameter b, comprises a classical action and one-loop determinants which are obtained by evaluating Gaussian integrals over vector and chiral multiplets around each saddle point.

The main purpose of this thesis is to give a detailed description of supersymmetric vortex loops based on the paper [6]. Vortex loops play an important role in the study of 3D gauge theories like Wilson loops. They are one-dimensional defects in 3D gauge theories typically defined by a singular behavior of the gauge field A:

$$A \sim \beta \mathrm{d}\varphi,$$

where φ is the angle coordinate that goes around the vortex worldline and the parameter β, called vorticity, can be gauge-rotated to be in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Based on this definition, supersymmetric vortex loops were first studied in [7] for ABJM model [8]. Exact computation of their expectation values was performed by [9, 10], but so far it has been mostly restricted to abelian gauge theories. Moreover, the results seem to indicate that vortex loops in abelian $\mathcal{N} = 2$ gauge theories are trivial; namely, as far as supersymmetric observables are concerned, they are equivalent to the identity operator.

There is another definition for the vortex loops based on the idea that the loop supports a quantum mechanics interacting with the 3D fields. A systematic identification in 3D $\mathcal{N} = 4$ theories was given in [11] using mirror symmetry [12] and type IIB brane construction [13–15]. However, generalization of their result to theories with less SUSY does not seem straightforward. Also, the correspondence between this definition and the previous definition based on singular gauge field is not fully clear yet.

The analysis of vortex loops in this thesis can be divided into two main parts. First, based on the definition in terms of singular gauge field, we give an exact formula for the path integral in the presence of vortex loops: the expectation value of a vortex loop on an ellipsoid is given by

$$\langle V_\beta \rangle = \frac{1}{|\mathcal{W}|} \int \mathrm{d}^r \hat{\sigma}\, e^{-S} \cdot \Delta^{\mathrm{v}}_{\text{1-loop}} \cdot \Delta^{\mathrm{c}}_{\text{1-loop}} \cdot V_\beta(\hat{\sigma}).$$

Comparing the integrand above with that of $Z_{S^3_b}$, we determine the function $V_\beta(\hat{\sigma})$ which encodes the effect due to the presence of a vortex loop.

As consistency check of our definition, we test this result against the known equivalence of Wilson and vortex loops in pure CS theory. In [16], Moore and Seiberg claimed that

$$V_\beta(C) \simeq W_\lambda(C) \qquad \text{for} \qquad \lambda = \frac{k\beta}{2},$$

where $W_\lambda(\mathrm{C})$ is a Wilson loop operator in a representation of the gauge group with the highest weight λ and k is the Chern-Simons level. The original proof of the equivalence [16] used the coadjoint orbit quantization for representing Wilson loops. It can actually be thought of as a prototypical example of a quantum mechanics on a loop interacting with the field theory in 3D space. By understanding the equivalence of the Wilson and vortex loops, we have made in [6] the first precise correspondence between the two definitions of vortex loops explained above.

In fact, by a naive comparison in $\mathcal{N} = 2$ CS theory we find there is an unwanted shift $\tilde{\rho}$ of parameters in the equivalence relation:

$$V_\beta(\widehat{\sigma}) \simeq W_\lambda(\widehat{\sigma}) \qquad \text{for} \qquad \lambda + \tilde{\rho} = \frac{k\beta}{2}\,.$$

This was already pointed out in [17]. At the end of Chapter 3 we propose a resolution which relates the shift to the global anomaly in $\mathcal{N} = 2$ SUSY quantum mechanics [18].

Second, we extend the correspondence of the two definitions of vortex loops to a wider class of $\mathcal{N} = 2$ theories. For this purpose we will focus on vortex loops in $U(N)$ gauge theories. We begin in Chapter 4 by developing the description of coadjoint orbit quantum mechanics as quiver gauged linear sigma models (GLSMs) of the kind studied in [19, 20]. The index $I(\hat{\sigma})$ of the GLSM is computed by JK residue prescription [18] and we can confirm the correspondence of the two description through the check of the relation $I(\widehat{\sigma}) = V_\beta(\widehat{\sigma})$. We also identify the extensions of these GLSMs that account for the addition of various matter chiral multiplets on the vortex background. This will be done for the matters in the adjoint, fundamental and anti-fundamental representations of $U(N)$.

As another extension, we study 1/2 BPS[1] vortex loops in $\mathcal{N} = 4$ theories. We present all the possible boundary conditions for $\mathcal{N} = 4$ multiplets in order for the vortex loop to preserve 1/2 SUSY. We also identify the corresponding worldline quantum mechanics with 1D $\mathcal{N} = 4$ supersymmetry.

Organization of this thesis

This thesis starts with a review of $\mathcal{N} = 2$ SUSY gauge theories in Chapter 1. Some preparations necessary for dealing with curved manifolds are also introduced there. In Chapter 2, exact formulae for the partition function and the expectation value of vortex the loop are derived using localization techniques. In Chapter 3, we then test this result against the known equivalence of Wilson and vortex loops in pure CS theory. GLSM descriptions are introduced in Chapter 4. The second half of this chapter, we extend the GLSMs to describe vortex loops in 3D theories with various matter chiral multiplets. Vortex loops in $\mathcal{N} = 4$ theories are studied in Chapter 5 where our construction reproduces some of the GLSMs for vortex loops that are identified in [11]. We conclude in Chapter 6 with the summary and discussions.

[1]The Bogomolny-Prasad-Sommerfield (BPS) bound, named after Evgeny B. Bogomolny [21], M. K. Prasad, and Charles M. Sommerfield [22], is originally a lower bound for the mass of a monopole that is set by its charge. When the bound is satisfied, the field equation simplifies, and this state is referred to as "saturated". In theories with supersymmetry, states or objects that satisfy similar bounds often preserve a portion of SUSY. Therefore, the terms "BPS" and "SUSY-preserving" refer to the same concept.

Chapter 1

Supersymmetric gauge theories

In this chapter, we consider how to realize supersymmetric field theories on certain three-dimensional manifolds. Our main interest in this thesis will be $\mathcal{N} = 2$ supersymmetric field theory on 3D ellipsoid, which is a squashed S^3. For this purpose, we first construct the theory on flat three-dimensional space. After some preparations to describe quantities on the curved geometry, namely basic formulation of general relativity, we generalize our construction to Riemannian manifolds. The naive general covariantization of the flat space theories does not possess supersymmetry, but adding appropriate non-minimal couplings can keep the theories supersymmetric. The condition that the theory admits one or several supersymmetries translates into an equation that the supersymmetric transformation parameters must satisfy, called the *Killing spinor equation* [23, 24]. In the first part of this chapter, we will explain this construction and present supersymmetric Lagrangians.

In the latter part of this chapter, we will introduce supersymmetric vortex operators, which are one-dimensional defect operators in three-manifolds. Naive volume integrals of Lagrangians may be divergent if such operators are inserted, because of a singular behavior of the gauge field. The divergence is regularized by removing a tubular neighborhood of the operator and adding specific boundary terms to the action [9, 10].

1.1 3D $\mathcal{N} = 2$ field theories on $\mathbb{R}^3$

In this paper we will consider three-dimensional $\mathcal{N} = 2$ supersymmetric field theories. The three-dimensional $\mathcal{N} = 2$ supersymmetry algebra(superalgebra) has four real supercharges, $Q_\alpha, \bar{Q}_\alpha (\alpha = 1, 2)$. This is the same amount of supersymmetry as in four-dimensional $\mathcal{N} = 1$ field theories, and many properties of the three-dimensional superalgebra can be deduced by reduction from four dimensions. Let us first describe some basic properties of these theories in flat three-dimensional space with Euclidean signature, in preparation for studying them on curved backgrounds later. For more detail, see, e.g., [25–27].

The $\mathcal{N} = 2$ SUSY algebra consists of the supercharges, satisfying the algebra:

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}^\alpha, \bar{Q}^\beta\} = 0, \quad \{Q_\alpha, \bar{Q}^\beta\} = (\gamma^a)_\alpha{}^\beta P_a + i\delta_\alpha{}^\beta Z \tag{1.1.1}$$

Here P_a is the momentum, Z is a real central charge. We choose the three-dimensional γ-matrices to be the Pauli matrices:

$$(\gamma^a)_\alpha{}^\beta = \{\sigma^1, \sigma^2, \sigma^3\} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{1.1.2}$$

where $a = 1, 2, 3$ is flat index. They satisfy

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}, \quad \gamma^{ab} \equiv \frac{1}{2}\left[\gamma^a, \gamma^b\right] = i\varepsilon^{abc}\gamma^c, \tag{1.1.3}$$

where ε^{abc} is the totally anti-symmetric symbol with $\varepsilon^{123} = 1$.

For a Lie group G with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, an $\mathcal{N} = 2$ supersymmetric field theory with gauge group G is made of a vectormultiplet $V = (A_m, \sigma, \lambda, \bar{\lambda}, D)$ in the adjoint representation of $\mathfrak{g}$, and by a chiral multiplet $\Phi = (\phi, \psi, F)$ and an anti-chiral multiplet $\bar{\Phi} = (\bar{\phi}, \bar{\psi}, \bar{F})$ in some representation $\mathbf{R}$ and $\bar{\mathbf{R}}$ of $\mathfrak{g}$, respectively.

Given such supermultiplets, one can write the supersymmetric transformation rules and SUSY-invariant Lagrangians at least on flat space. In fact, most of the formulae of 3D $\mathcal{N} = 2$ supersymmetric field theory can be obtained by a dimensional reduction of 4D $\mathcal{N} = 1$ theory for which all the basic formula are given in the textbook by Wess-Bagger [28]. Note that whereas the [28] is written in superfield notation, we prefer to use component fields since it is more convenient to deal with the theory on curved space.

1.1.1 Chiral multiplets

A chiral multiplet Φ consists of a complex scalar ϕ, a complex spinor ψ, and an auxiliary complex scalar F. We denote the generator of supersymmetric transformation as

$$\boldsymbol{Q} = \xi Q + \bar{\xi}\bar{Q}, \tag{1.1.4}$$

where $\xi, \bar{\xi}$ are constant spinors called supersymmetric parameters. In many papers, including the references cited above, SUSY operators are defined as Grassmann-even operators. But we define it to be Grassmann-odd: supersymmetric parameters $\xi, \bar{\xi}$ are regarded as Grassmann-even spinors[2]. We denote the bilinears of spinors by

$$\xi\psi \equiv \xi_\alpha C^{\alpha\beta}\psi_\beta, \qquad \xi\gamma^a\psi \equiv \xi_\alpha C^{\alpha\beta}(\gamma^a)_\beta{}^\rho\psi_\rho, \tag{1.1.5}$$

where $C^{\alpha\beta}$ is an anti-symmetric matrix with $C^{12} = -C^{21} = 1$. From now on, the indices for 2-component spinors are always suppressed except needed.

The supersymmetric transformation rule on flat space for a chiral multiplet Φ (not having gauge charges) is given by

$$\boldsymbol{Q}\phi = \xi\psi, \quad \boldsymbol{Q}\psi = i\gamma^a\partial_a\phi\bar{\xi} + F\xi, \quad \boldsymbol{Q}F = i\bar{\xi}\gamma^a\partial_a\psi. \tag{1.1.6}$$

[2] Our notation and spinor conventions will mostly follow [6, 29].

The CPT conjugate of Φ is an anti-chiral multiplet $\bar{\Phi} = (\bar{\phi}, \bar{\psi}, \bar{F})$, valued in the conjugate representation $\bar{\mathbf{R}}$ of $\mathfrak{g}$, with the supersymmetric transformations:

$$\boldsymbol{Q}\bar{\phi} = \bar{\xi}\bar{\psi}, \quad \boldsymbol{Q}\bar{\psi} = i\gamma^a\partial_a\bar{\phi}\xi + \bar{F}\bar{\xi}, \quad \boldsymbol{Q}\bar{F} = i\xi\gamma^a\partial_a\bar{\psi}. \tag{1.1.7}$$

One can compute the square of $\boldsymbol{Q}$, for example, by acting $\boldsymbol{Q}$ on ϕ twice:

$$\begin{aligned}\boldsymbol{Q}^2\phi =&\boldsymbol{Q}(\xi\psi)\\ =&\xi\left(i\gamma^a\partial_a\phi\bar{\xi} + F\xi\right)\\ =&i\bar{\xi}\gamma^a\xi\partial_a\phi.\end{aligned} \tag{1.1.8}$$

Here, the second term in the second line is zero due to the spinor multiplication rule $\xi\xi' = -\xi'\xi$. Similarly, the action of $\boldsymbol{Q}^2$ on the other fields yields

$$\boldsymbol{Q}^2\phi = iv^a\partial_a\phi, \qquad \boldsymbol{Q}^2\psi = iv^a\partial_a\psi, \qquad \boldsymbol{Q}^2F = iv^a\partial_aF, \tag{1.1.9}$$
$$\boldsymbol{Q}^2\bar{\phi} = iv^a\partial_a\bar{\phi}, \qquad \boldsymbol{Q}^2\bar{\psi} = iv^a\partial_a\bar{\psi}, \qquad \boldsymbol{Q}^2\bar{F} = iv^a\partial_a\bar{F}, \tag{1.1.10}$$

where $v^a \equiv \bar{\xi}\gamma^a\xi$. Therefore the square of $\boldsymbol{Q}$ acts as

$$\boldsymbol{Q}^2 = iv^a\partial_a \tag{1.1.11}$$

on all the fields. In fact, as we will see later, the square of $\boldsymbol{Q}$ acts as a sum of bosonic symmetries. In the present case, the right-hand side of (1.1.11) contains only the spacetime symmetry. In terms of supersymmetric algebra (1.1.1), it corresponds to $Z = 0$. The SUSY invariant Lagrangian consisting only of $\Phi, \bar{\Phi}$ is given by

$$\mathcal{L}_{\text{mat}} = \partial_a\bar{\phi}\,\partial^a\phi - i\bar{\psi}\,\partial\!\!\!/\psi + F\bar{F}. \tag{1.1.12}$$

1.1.2 Vectormultiplets

A vectormultiplet corresponds to a real superfield V which is subject to a kind of gauge transformation. In the so-called Wess-Zumino (WZ) gauge, the vectormultiplet consists of a vector A_m, a real scalar σ, a pair of complex fermions $\lambda, \bar{\lambda}$, and an auxiliary real scalar D which are all $\mathfrak{g} = \mathrm{Lie}(G)$ valued. They transform under supersymmetry as

$$\begin{aligned}\boldsymbol{Q}A_a =& -\frac{i}{2}(\bar{\xi}\gamma_a\lambda + \xi\gamma_a\bar{\lambda}),\\ \boldsymbol{Q}\sigma =& \frac{1}{2}(\xi\bar{\lambda} - \bar{\xi}\lambda),\\ \boldsymbol{Q}\lambda =& \frac{1}{2}\gamma^{ab}\xi F_{ab} - \xi D - i\not{D}\sigma\cdot\xi,\\ \boldsymbol{Q}\bar{\lambda} =& \frac{1}{2}\gamma^{ab}\bar{\xi}F_{ab} + \bar{\xi}D + i\not{D}\sigma\cdot\bar{\xi},\\ \boldsymbol{Q}D =& \frac{i}{2}(\xi\not{D}\bar{\lambda} - \bar{\xi}\not{D}\lambda) + \frac{i}{2}(\xi[\sigma,\bar{\lambda}] + \bar{\xi}[\sigma,\lambda]),\end{aligned} \tag{1.1.13}$$

where

$$F_{ab} \equiv \partial_aA_b - \partial_bA_a - i[A_a, A_b], \qquad D_a\sigma \equiv \partial_a\sigma - i[A_a, \sigma]. \tag{1.1.14}$$

This also modifies the supersymmetry transformations of the chiral multiplet by terms involving the vectormultiplet fields, which includes the replacements $\partial_a \to D_a$:

$$\begin{aligned} \boldsymbol{Q}\phi &= \xi\psi, & \boldsymbol{Q}\bar{\phi} &= \bar{\xi}\bar{\psi}, \\ \boldsymbol{Q}\psi &= i(\not{D}\phi + \sigma\phi)\bar{\xi} + F\xi, & \boldsymbol{Q}\bar{\psi} &= i(\not{D}\bar{\phi} + \bar{\phi}\sigma)\xi + \bar{F}\bar{\xi}, \\ \boldsymbol{Q}F &= i\bar{\xi}(\not{D}\psi - \sigma\psi) - i\bar{\xi}\bar{\lambda}\phi, & \boldsymbol{Q}\bar{F} &= i\xi(\not{D}\bar{\psi} - \bar{\psi}\sigma) + i\xi\bar{\phi}\lambda \end{aligned} \tag{1.1.15}$$

where

$$\begin{aligned} D_a\phi &\equiv \partial_a\phi - iA_a\phi, & D_a\bar{\phi} &\equiv \partial_a\bar{\phi} + i\bar{\phi}A_a, \\ D_a\psi &\equiv \partial_a\phi - iA_a\psi, & D_a\bar{\psi} &\equiv \partial_a\bar{\psi} + i\bar{\psi}A_a. \end{aligned} \tag{1.1.16}$$

As in the previous section, one can compute the square of $\boldsymbol{Q}$ by acting $\boldsymbol{Q}$ twice on ϕ. The results is

$$\begin{aligned} \boldsymbol{Q}^2\phi &= \boldsymbol{Q}(\xi\psi) \\ &= iv^a\partial_a\phi + \Sigma\phi, \end{aligned} \tag{1.1.17}$$

where $\Sigma = v_a A_a - i\bar{\xi}\xi\sigma$. Similarly, acting on the other fields, which are components of a (anti-)chiral multiplet, one obtains

$$\begin{aligned} \boldsymbol{Q}^2\phi &= iv^a\partial_a\phi + \Sigma\phi, & \boldsymbol{Q}^2\bar{\phi} &= iv^a\partial_a\bar{\phi} - \bar{\phi}\Sigma, \\ \boldsymbol{Q}^2\psi &= iv^a\partial_a\psi + \Sigma\psi, & \boldsymbol{Q}^2\bar{\psi} &= iv^a\partial_a\bar{\psi} - \bar{\psi}\Sigma, \\ \boldsymbol{Q}^2F &= iv^a\partial_a F + \Sigma F, & \boldsymbol{Q}^2\bar{F} &= iv^a\partial_a\bar{F} - \bar{F}\Sigma, \end{aligned} \tag{1.1.18}$$

and for the fields in a vectormultiplet one obtains

$$\begin{aligned} \boldsymbol{Q}^2 A_a &= iv^b\partial_b A_a - iD_a\Sigma, \\ \boldsymbol{Q}^2\sigma &= iv^a\partial_a\sigma + [\Sigma, \sigma], \\ \boldsymbol{Q}^2\lambda &= iv^a\partial_a\lambda + [\Sigma, \lambda], \\ \boldsymbol{Q}^2\bar{\lambda} &= iv^a\partial_a\bar{\lambda} + [\Sigma, \bar{\lambda}], \\ \boldsymbol{Q}^2 D &= iv^a\partial_a D + [\Sigma, D]. \end{aligned} \tag{1.1.19}$$

As noted before, the square of $\boldsymbol{Q}$ generates a sum of bosonic symmetry transformations

$$\boldsymbol{Q}^2 = iv^a\partial_a + \mathsf{Gauge}_\Sigma \tag{1.1.20}$$

where, for example,

$$\mathsf{Gauge}_\Sigma\Phi = \Sigma_{\mathbf{R}}\Phi, \quad \mathsf{Gauge}_\Sigma\bar{\Phi} = -\bar{\Phi}\Sigma_{\mathbf{R}}, \quad \mathsf{Gauge}_\Sigma\Phi_{\mathbf{adj}} = [\Sigma, \Phi_{\mathbf{adj}}], \tag{1.1.21}$$

for Φ in $\mathbf{R}$, $\bar{\Phi}$ in $\bar{\mathbf{R}}$, and V in the adjoint representation. It corresponds to $Z = \Sigma$ in terms of the supersymmetric algebra (1.1.1).

1.1.3 Lagrangians

Let us list the building blocks of supersymmetric Lagrangians. First, we consider a chiral multiplet Φ coupled to a vectormultiplet V. The standard kinetic term for the chiral multiplet fields reads:

$$\begin{aligned}\mathcal{L}_{\rm mat} =& D_a\bar{\phi}D_a\phi + \bar{\phi}\sigma^2\phi - i\bar{\phi}D\phi - \frac{i}{2}\bar{\psi}\gamma^a D_a\psi + \frac{i}{2}D_a\bar{\psi}\gamma^a\psi + i\bar{\psi}\sigma\psi \\ &+ \bar{F}F + i\bar{\psi}\bar{\lambda}\phi - i\bar{\phi}\lambda\psi.\end{aligned} \tag{1.1.22}$$

This Lagrangian is also expressed as follows by using the superspace formalism.

$$\mathcal{L}_{\rm mat} = \int \mathrm{d}\theta^4 \mathcal{K}(\Phi,\bar{\Phi},V), \quad \mathcal{K}(\Phi,\bar{\Phi},V) = \bar{\Phi}e^{-V}\Phi. \tag{1.1.23}$$

We can also consider the so-called F-term or the superpotential term for chiral multiplets:

$$\mathcal{L}_{\rm pot} = \int \mathrm{d}\theta^2 W(\Phi_i) + \text{c.c.} = \frac{\partial W}{\partial\Phi_i}F_i + \frac{\partial^2 W}{\partial\Phi_i\partial\Phi_j}\psi_i\psi_j + \text{c.c.}, \tag{1.1.24}$$

with a holomorphic function $W(\Phi_i)$ called the super potential.

For the vectormultiplet, there are two choices for the kinetic term. One is a supersymmetric extension of the Chern-Simons (CS) term[3]:

$$\mathcal{L}_{\rm CS} = \frac{ik}{4\pi}\,\mathrm{Tr}\left[\varepsilon^{abc}\left(A_a\partial_b A_c - \frac{2i}{3}A_aA_bA_c\right) - \bar{\lambda}\lambda + 2D\sigma\right], \tag{1.1.27}$$

where k is called the CS level and "Tr" stands for the *standard trace*[4]. These kinetic terms $\mathcal{L}_{\rm mat}$, $\mathcal{L}_{\rm CS}$ preserve scale invariance classically[5]. The other choice of kinetic term for the vectormultiplet fields is the Yang-Mills Lagrangian

$$\mathcal{L}_{\rm YM} = \frac{1}{g^2}\,\mathrm{Tr}\left[\frac{1}{2}F_{ab}^2 + (D_a\sigma)^2 + D^2 + \frac{i}{2}\bar{\lambda}\gamma^a D_a\lambda - \frac{i}{2}D_a\bar{\lambda}\gamma^a\lambda - i\bar{\lambda}[\sigma,\lambda]\right]. \tag{1.1.28}$$

[3]The bosonic CS theory is defined by

$$\begin{aligned}\int \mathcal{D}\hat{A}\exp\left[\frac{ik}{4\pi}\int \mathrm{Tr}\left(\hat{A}d\hat{A} + \frac{2}{3}\hat{A}^3\right)\right] &= \int \mathcal{D}A\exp\left[\frac{-ik}{4\pi}\int \mathrm{Tr}\left(AdA - \frac{2i}{3}A^3\right)\right] \qquad (1.1.25)\\ &\equiv \int \mathcal{D}A\exp\left(-S_{\rm CS}[A]\right) \qquad (1.1.26)\end{aligned}$$

with $\hat{A}$ anti-Hermite and $A = -i\hat{A}$ Hermite.

[4]For a simple group G, the trace in any representation R of a Lie group G gives an invariant bilinear form $\mathrm{Tr}(XY)$ for the Lie algebra elements X, Y. Here "Tr" stands for the *standard trace* which is defined by

$$\mathrm{Tr}(XY) = \frac{1}{2T_R}\,\mathrm{tr}_R(XY),$$

where $2T_R$ is knows as the Dynkin index. For example, $T_R = \frac{1}{2}, 1, \frac{1}{2}$ for the representation $R = \mathbb{C}^n, \mathbb{R}^n, \mathbb{C}^{2n}$ of $SU(n), SO(n), Sp(n)$.

[5]In fact, $\mathcal{N} = 2$ field theory defined by $\mathcal{L}_{\rm mat}$ with the CS term $\mathcal{L}_{\rm CS}$ preserve scale invariance at the quantum level [30].

1.1.4 Real mass and FI parameters

When we consider the theory that has a non-trivial continuous global symmetry group G_F, it is useful to turn on a background vectormultiplet V_F. We should think of this background field as classical, and it takes fixed values that appear as the parameters in Lagrangians. In order to preserve the supersymmetry, a background gauge field $V_F = (A_a^{(F)}, \sigma^{(F)}, \lambda^{(F)}, \bar{\lambda}^{(F)}, D^{(F)})$ takes values such that the supersymmetric transformation of gaugino vanishes: $\boldsymbol{Q}\lambda^{(F)} = \boldsymbol{Q}\bar{\lambda}^{(F)} = 0$. In flat space, one should take

$$A_a^{(F)} = D^{(F)} = 0, \quad \sigma^{(F)} = m_F \in \mathfrak{g}_F. \tag{1.1.29}$$

For a chiral multiplet with charge q under a global $U(1)$ symmetry, after turning on the background vectormultiplet field, one finds additional terms in the action:

$$\mathcal{L}_{\text{mat}} = \cdots + \bar{\phi}(qm_F)^2\phi + i\bar{\psi}qm_F\psi. \tag{1.1.30}$$

The real parameter m_F is called *real mass* in 3D, since qm in (1.1.30) corresponds to a mass for both ϕ and ψ. This also modifies the supersymmetric transformation (1.1.15), for example,

$$\boldsymbol{Q}\psi = \cdots + iqm_F\phi\bar{\xi}, \qquad \boldsymbol{Q}F = \cdots - iqm_F\bar{\xi}\psi. \tag{1.1.31}$$

If a theory has a $U(1)$ gauge symmetry, one can define a current:

$$J^a_{\text{top}} = \tilde{F}^a \equiv \frac{1}{2}\varepsilon^{abc}F_{bc}, \tag{1.1.32}$$

which is conserved due to Bianchi identity. The corresponding global symmetry, $U(1)_T$, is called a "topological symmetry". The charged objects of this symmetry are monopole operators. To gauge this symmetry with a vectormultiplet $\tilde{V} = (\tilde{A}_a, \tilde{\sigma}, \tilde{\lambda}, \bar{\tilde{\lambda}}, \tilde{D})$, one adds the supersymmetric extension of coupling term $\tilde{A}_a J^a_{\text{top}}$:

$$\frac{i}{2\pi}(\varepsilon^{abc}\tilde{A}_a\partial_b A_c + D\tilde{\sigma} + \sigma\tilde{D} - \bar{\lambda}\tilde{\lambda} + \bar{\tilde{\lambda}}\lambda), \tag{1.1.33}$$

which is a mixed Chern-Simons term. If one regards $\tilde{V}$ as the background vectormultiplet and turns on a constant value for the scalar $\tilde{\sigma} = \zeta$, one obtains a Fayet-Iliopoulos (FI) term:

$$\mathcal{L}_{\text{FI}} = \frac{i\zeta}{2\pi}D. \tag{1.1.34}$$

1.1.5 R-symmetry

The $\mathcal{N} = 2$ algebra has a $U(1)$ symmetry rotating the supercharges

$$Q \to e^{i\alpha}Q, \qquad \bar{Q} \to e^{-i\alpha}\bar{Q}, \tag{1.1.35}$$

which is called $U(1)_R$ symmetry or simply R-symmetry. The supersymmetric parameters $\xi, \bar{\xi}$ have R-charge $+1$, -1 respectively, so they are rotated as:

$$\xi \to e^{-i\alpha}\xi, \qquad \bar{\xi} \to e^{i\alpha}\bar{\xi}. \tag{1.1.36}$$

One can assign the R-charges to the component fields in a single supermultiplet so that the SUSY transformation (1.1.13), (1.1.15) preserves the R-charge.

$$\begin{aligned}
&\phi \to e^{-ir\alpha}\phi, && \bar{\phi} \to e^{ir\alpha}\bar{\phi}, \\
&\psi \to e^{-i(r-1)\alpha}\psi, && \bar{\psi} \to e^{i(r-1)\alpha}\bar{\psi}, \\
&F \to e^{-i(r-2)\alpha}F, && \bar{F} \to e^{i(r-2)\alpha}\bar{F}.
\end{aligned} \tag{1.1.37}$$

If the $U(1)_R$ symmetry is gauged by V_m, the derivatives acting on the fields with charge r should be covariantized as follows.

$$\partial_m \to \partial_m - irV_m. \tag{1.1.38}$$

1.2 3D $\mathcal{N}=2$ field theories on curved manifold

We have reviewed the SUSY field theory on flat space so far. This section aims to realize the SUSY field theory on a certain class of Riemannian three-manifolds $\mathcal{M}_3$.

In general, the background metric breaks supersymmetry completely. Indeed, supersymmetry is an extension of the Poincaré symmetry group, the isometry group of flat space, which is also completely broken on an arbitrary manifold with a generic metric. On the other hand, some $\mathcal{M}_3$ admit Killing vector fields v^m which generate non-trivial isometries. Similarly, some may also admit (generalized) Killing spinors, denoted by $\xi, \bar{\xi}$, which generate curved-space supersymmetries. The Killing vectors and spinors then generate a "rigid supersymmetric algebra" in curved space.

Whether the SUSY theories are realized on $\mathcal{M}_3$ translates into the question whether the Killing spinors can be defined on $\mathcal{M}_3$. On Riemannian manifold, the SUSY parameters $\xi, \bar{\xi}$ which have been constant spinors on flat space are no longer constants, but are the Killing spinors which are understood as solutions to the *Killing spinor equation* on the manifold, possibly with additional background fields. According to Festuccia and Seiberg [23], this equation for curved-space supersymmetry arises from the rigid limit of supergravity. In this paper, we are mainly interested in 3D $\mathcal{N}=2$ field theories with an $U(1)_R$ symmetry. Such theories can be coupled to the 3D $\mathcal{N}=2$ "new-minimal" supergravity.

To have a rigid supersymmetry, one assumes all the fields in supergravity to take some classical values, and in particular all the fermionic fields, such as gravitinos, are set to zero. In addition, the bosonic fields are determined from that the $\boldsymbol{Q}$-variation of fermions are zero. The requirement of the vanishing is nothing but the Killing spinor equation. Although one could find the suitable background fields on each three-manifold as have been done on S^3 [1,3,4] and generalized to ellipsoid [5,31,32], we will follow the systematic way [24] based on the Festuccia-Seiberg approach [23] to determine the background fields. See also review [25,26,33].

In the first half of this section, we make some preparations for dealing with curved manifolds. In particular, we describe how the various quantities on $\mathcal{M}_3$ behave under the general coordinate and the local Lorentz transformations. After that we will discuss the Killing spinor equation, and then explicitly derive the Killing spinor and the background fields on some specific Riemannian three-manifolds $\mathcal{M}_3$.

1.2.1 The general coordinate transformation

Here we summarize some standard facts about general relativity. First, consider a three-dimensional Riemannian manifold $\mathcal{M}_3$ with metric g_{mn}:

$$ds^2 = g_{mn}(x)\mathrm{d}x^m\mathrm{d}x^n. \tag{1.2.1}$$

An object $A = A_m(x)\mathrm{d}x^m$ on $\mathcal{M}_3$ which transforms under general coordinate transformation as

$$A_m(x)\mathrm{d}x^m = \tilde{A}_m(\tilde{x})\mathrm{d}\tilde{x}^m, \tag{1.2.2}$$

is called a 1-form. The metric g_{mn} or its inverse g^{mn} are used to lower or raise vector indices, for instance $A_m g^{mn} = A^n$ and $A^m g_{mn} = A_n$. For an infinitesimal general coordinate transformation

$$x^m \to \tilde{x}^m = x^m - v^m(x), \tag{1.2.3}$$

the difference between a tilded and a non-tilded vector fields(1.2.2) defines the Lie derivative $\pounds_v$:

$$\begin{aligned}\pounds_v A_m &\equiv \tilde{A}_m(\tilde{x}) - A_m(\tilde{x})\\ &= v^n\partial_n A_m + \partial_m v^n \cdot A_n\\ &= v^m\nabla_m A_n + \nabla_n v^m \cdot A_m.\end{aligned} \tag{1.2.4}$$

Here the covariant derivative ∇_m is defined with the *affine connection* Γ^l_{mn} as follows.

$$\begin{aligned}\nabla_m A_n &= \partial_m A_n - \Gamma^l_{mn}A_l,\\ \nabla_m A^n &= \partial_m A^n + \Gamma^n_{ml}A^l.\end{aligned} \tag{1.2.5}$$

The covariant derivatives of the vector fields $\nabla_m A_n,\ \nabla_m A^n$ behave as tensor fields, whereas the partial derivatives of vectors $\partial_m A_n,\ \partial_m A^n$ do not. The affine connection is determined from that it is symmetric and metric compatible:

$$\Gamma^l_{mn} = \Gamma^l_{nm}, \qquad \nabla_m g_{nl} = 0. \tag{1.2.6}$$

In general relativity, these properties come as a consequence of the Einstein's principle of equivalence. It is easy to show that (1.2.6) imply

$$\Gamma^l_{mn} = \frac{1}{2}g^{lq}\left(\partial_m g_{qn} + \partial_n g_{mq} - \partial_q g_{mn}\right). \tag{1.2.7}$$

The Riemann tensor $R^p{}_{qmn}$ is defined by

$$\begin{aligned}\left[\nabla_m, \nabla_n\right] A^p &= R^p{}_{qmn}A^q,\\ \left[\nabla_m, \nabla_n\right] A_p &= -R^q{}_{pmn}A_q,\end{aligned} \tag{1.2.8}$$

where

$$R^p{}_{qmn} \equiv \partial_m \Gamma^p_{nq} - \partial_n \Gamma^p_{mq} + \Gamma^p_{ms}\Gamma^s_{nq} - \Gamma^p_{ns}\Gamma^s_{mq}. \tag{1.2.9}$$

It is easy to see that $R_{pqmn} = g_{ps}R^s{}_{qmn}$ is antisymmetric in the indices p, q and likewise in m, n, while it is symmetric under the exchange of the pairs (p, q) and (m, n). The Ricci tensor R_{mn} and the Ricci scalar R are defined as follows.

$$\begin{aligned} R_{mn} &\equiv R^p{}_{mpn} = g^{pq}R_{pmqn}, \\ R &\equiv g^{mn}R_{mn}. \end{aligned} \tag{1.2.10}$$

If a vector v^m satisfies the condition:

$$0 = \pounds_v g_{mn} = \nabla_m v_n + \nabla_n v_m, \tag{1.2.11}$$

the vector v^m is called a *Killing vector* on $\mathcal{M}_3$. Thus, the tensor $\nabla_m v_n$ is antisymmetric in the indices m, n if v^m is a Killing vector.

Note that the definition of the Lie derivative above is compatible with the usual mathematical definition of that for a differential form:

$$\pounds_v \cdot \equiv \imath_v(\mathrm{d}\,\cdot\,) + \mathrm{d}\,(\imath_v\,\cdot\,), \tag{1.2.12}$$

where d and $\imath$ stand for the external derivative and the interior product. For example, for a 1-form $A = A_m \mathrm{d}x^m$,

$$\begin{aligned} \pounds_v A &= \mathrm{d}(\imath_v A) + \imath_v(\mathrm{d}A) \\ &= \partial_m(v^n A_n)\mathrm{d}x^m + v^m \partial_m A_n \mathrm{d}x^n - v^m \partial_n A_m dx^n \\ &= (\partial_n v^m A_m + v^m \partial_m A_n)\,\mathrm{d}x^m. \end{aligned} \tag{1.2.13}$$

1.2.2 The local Lorentz transformation

For an arbitrary given point P on a Riemannian manifold $\mathcal{M}_3$ with g_{mn}, it is possible to find a locally flat coordinate system near P. Such a coordinate system is called a local Lorentz frame. One can introduce an orthonormal set of 1-forms e^a_m such that

$$e^a_m e^{bm} = g^{mn} e^a_m e^b_n = \eta^{ab}, \tag{1.2.14}$$

where $\eta^{ab} = \mathrm{diag}(+1, +1, +1)$ is the (inverse of) Euclidean metric, m, n are curved indices and $a, b = 1, 2, 3$ are flat indices[6]. Alteratively, one can write the metric as an inner product of e^a_m:

$$g_{mn} = e^a_m e^b_n \eta_{ab}. \tag{1.2.15}$$

A set of vectors e^a_m are called a *vielbein* that means "many legs" in German. In the present case, they are also called a *dreibein* since e^a_m are a tripod. Given a metric g_{mn} on $\mathcal{M}_3$, one can always

[6] In what follows, the upper and lower flat indices might not always be placed in accordance with the Einstein's rule. But it will not cause as we work on Euclidean signature. On the other hand, for curved indices the distinction of that is important.

construct a dreibein e_m^a satisfying (1.2.15). In fact, the dreibein is not uniquely determined, but is transformed under local Lorentz transformations:

$$e_m^a(x) \to \tilde{e}_m^a(x) = \Lambda^a{}_b(x) e_m^b(x). \tag{1.2.16}$$

The metric g_{mn} is invariant under this transformation.

$$\begin{aligned} g_{mn} \to \tilde{g}_{mn} &= \eta_{ab} \tilde{e}_m^a \tilde{e}_n^b \\ &= \eta_{ab} \Lambda^a{}_c \Lambda^b{}_d e_m^c e_d^b \\ &= \eta_{cd} e_m^c e_n^d, \end{aligned} \tag{1.2.17}$$

where an argument x was suppressed. The third equality $\eta_{ab}\Lambda^a{}_c\Lambda^b{}_d = \eta_{cd}$ means that η_{ab} is a local Lorentz invariant tensor. A vector field with a flat index such as A_a, A^a is in a vector representation of the local Lorentz group. Likewise, a spinor field Ψ is in a spinor representation of that.[7] The covariant derivatives of these fields are defined with the *spin connection* Ω_m^{ab}:

$$\begin{aligned} \nabla_m A^a &= \partial_m A^a + \Omega_m^{ab} A^b \\ \nabla_m \Psi &= \partial_m \Psi + \frac{1}{4} \Omega_m^{ab} \gamma^{ab} \Psi. \end{aligned} \tag{1.2.18}$$

To determine Ω_m^{ab} one uses the fact that e_n^a is covariantly constant like g_{mn}, namely $\nabla_m e_n^a = 0$. In fact, all we need is its antisymmetric part, which can be expressed in terms of differential forms as

$$0 = \nabla e^a = \mathrm{d}e^a + \Omega^{ab} \wedge e^b, \tag{1.2.19}$$

where we used $e^a = e_m^a \mathrm{d}x^m$, $\Omega^{ab} = \Omega_m^{ab} \mathrm{d}x^m$. The Riemann tenser with local Lorentz indices is defined by

$$\begin{aligned} \left[\nabla_m, \nabla_n\right] A^a &= R_{mn}^{ab} A^b, \\ \left[\nabla_m, \nabla_n\right] \psi &= \frac{1}{4} R_{mn}^{ab} \gamma^{ab} \psi, \end{aligned} \tag{1.2.20}$$

where R_{mn}^{ab} is expressed as

$$R_{mn}^{ab} \equiv \partial_m \Omega_n^{ab} - \partial_n \Omega_m^{ab} + \Omega_m^{ac} \Omega_n^{cb} - \Omega_n^{ac} \Omega_m^{cb}. \tag{1.2.21}$$

One can also show $R_{mn}^{ab} = R_{pqmn} e^{pa} e^{qb}$ via

$$\Gamma^l{}_{mn} e_l^a = \partial_m e_n^a + \Omega_m^{ab} e_n^b. \tag{1.2.22}$$

Next, let us consider how a Lie derivative $\pounds_v$ acts on the fields with flat indices. We define $\pounds_v$ so that it satisfy

$$\pounds_v e_m^a = 0, \tag{1.2.23}$$

if and only if the vector v is a Killing vector. This, indeed, satisfies the condition(1.2.11):

$$\pounds_v (e_m^a e_n^a) = \pounds_v e_m^a \cdot e_n^a + e_m^a \cdot \pounds_v e_n^a = 0. \tag{1.2.24}$$

[7] A spinor is not a representation of general coordinate transformation.

We suppose that, in addition to ordinary terms (1.2.4), the Lie derivatives can be modified by the local Lorentz transformation with a parameter $\Theta^{ab}_{(v)}$. This parameter is determined by solving (1.2.23):

$$0 = \pounds_v e^a_m = v^n \nabla_n e^a_m + \nabla_n v^m \cdot e^a_n + \Theta^{ab}_{(v)} e^b_m. \tag{1.2.25}$$

One thus obtain the Lie derivative of the fields in various representation of the local Lorentz group [34]:

$$\begin{aligned}
\pounds_v e^a_m &= v^n \partial_n e^a_m + \partial_n v^m \cdot e^a_n + \left(v^n \Omega^{ab}_n + \Theta^{ab}_{(v)}\right) e^b_m, \\
\pounds_v V^a &= v^m \partial_m V^a + \left(v^n \Omega^{ab}_n + \Theta^{ab}_{(v)}\right) V^b, \\
\pounds_v \Psi &= v^m \partial_m \Psi + \frac{1}{4}\left(v^n \Omega^{ab}_n + \Theta^{ab}_{(v)}\right) \gamma^{ab}\Psi,
\end{aligned} \tag{1.2.26}$$

where

$$\Theta^{ab}_{(v)} = \nabla_{[m} v_{n]} e^{am} e^{bn}. \tag{1.2.27}$$

1.2.3 Killing spinors

As was noted at the beginning of this section, when we realize supersymmetric theories on a curved manifold, the SUSY parameters are no longer constant spinors, but must be the Killing spinors, the solutions of the Killing spinor equation. The idea is based on the fact that a given configuration of the supergravity background fields preserves rigid supersymmetry if and only if gravitino variations vanish for some choice of SUSY parameter [23]. Since our interest is in $\mathcal{N} = 2$ supersymmetric field theories on $\mathcal{M}_3$, the most general form of the Killing spinor equation [24] is

$$\begin{aligned}
D_m \xi &\equiv (\nabla_m - iV_m)\,\xi = \frac{i}{2}\gamma_m \kappa, \\
D_m \bar{\xi} &\equiv (\nabla_m + iV_m)\,\bar{\xi} = \frac{i}{2}\gamma_m \bar{\kappa},
\end{aligned} \tag{1.2.28}$$
$$\kappa \equiv \left(H - i\not{K}\right)\xi, \qquad \bar{\kappa} \equiv \left(H + i\not{K}\right)\bar{\xi},$$

where $\not{K} = \gamma^m K_m$ and K_m is a smooth, conserved vector field $\nabla_m K^m = 0$. Note that the fields V_m, H, K_m are the background fields, which could also be interpreted as component fields of a supergravity multiplet[8]. In particular, V_m is the gauge field for the $U(1)$ R-symmetry and $\xi, \bar{\xi}$ have R-charge $+1, -1$, respectively.

If there exists a pair of $\xi, \bar{\xi}$ spinors satisfying the Killing spinor equation (1.2.28), the spinors $\xi, \bar{\xi}$ give rise to a Killing vector $v = v^m \partial_m$ with $v^m = \bar{\xi}\gamma^a \xi$. It generates an isometry of $\mathcal{M}_3$ with the Riemann metric g_{mn}, in other words it satisfies

$$\pounds_v g_{mn} = \nabla_m v_n + \nabla_n v_m = 0. \tag{1.2.29}$$

In the following, we will present the explicit form of the Killing spinors and the suitable background supergravity fields on an ellipsoid which are used throughout this thesis. The Killing spinors on some other manifolds are derived in Appendix A.

[8] Note that our notation (V_m, H, K_m) corresponds to $(A_\mu - \frac{3}{2}V_\mu, iH, iV_\mu)$ in [24, 26], and to $(A^{(R)}_\mu, iH, iV_\mu)$ in [25, 33].

An ellipsoid is a squashed three-sphere(S^3), which is embedded in $\mathbb{R}^4$ as follows [5]:

$$\frac{x_1^2+x_2^2}{\tilde{l}^2}+\frac{x_3^2+x_4^2}{l^2}=1, \qquad \mathrm{d}s^2=\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2+\mathrm{d}x_4^2. \tag{1.2.30}$$

Here $b \equiv \sqrt{l/\tilde{l}}$ is called the *squashing parameter* which represents a measure of squashing. In particular, the ellipsoid goes back to (round) S^3 for $l=\tilde{l}$, namely $b=1$. The supersymmetric partition function on this ellipsoid with the embedding will be shown to depend on b in a nontrivial manner [5].

Note that this squashing preserves only a $U(1)\times U(1)$ subgroup of the $SU(2)\times SU(2)$ isometry of S^3. Another squashing that preserves $SU(2)\times U(1)$ symmetry was also discussed in [5], where it was shown that the partition function with this deformation dose not depend on b. However, the paper [32] found that a squashing with the $SU(2)\times U(1)$ symmetry also gives a b dependent partition function if the set of background fields is appropriately chosen. In that paper, they showed that the 3D theory on such a background can be obtained by a specific compactification of the 4D $\mathcal{N}=1$ theory on $S^3\times S^1$.

By moving from cartesian coordinates to polar coordinates by substituting (x_1,x_2,x_3,x_4) with $(\cos\theta\cos\varphi, \cos\theta\sin\varphi, \sin\theta\cos\tau, \sin\theta\sin\tau)$, a set of dreibein is expressed as follows.

$$e^1=f(\theta)\,\mathrm{d}\theta, \quad e^2=\tilde{l}\sin\theta\,\mathrm{d}\varphi, \quad e^3=l\cos\theta\,\mathrm{d}\tau, \quad f(\theta)=\sqrt{\tilde{l}^2\sin^2\theta+l^2\cos^2\theta}. \tag{1.2.31}$$

The coordinates φ,τ correspond to rotations within (x^1,x^2) and (x^3,x^4)-planes, whereas θ takes values $0\le\theta\le\pi/2$. The Killing spinor equation has the following solutions:

$$\xi=e^{\frac{i}{2}(\varphi+\tau)}\begin{pmatrix}\cos\frac{\theta}{2}\\ i\sin\frac{\theta}{2}\end{pmatrix}, \qquad \bar{\xi}=e^{-\frac{i}{2}(\varphi+\tau)}\begin{pmatrix}i\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}, \tag{1.2.32}$$

if the background fields V_m, H, K_m take the following form.

$$V=\frac{1}{2}\left(1-\frac{\tilde{l}}{f}\right)\mathrm{d}\varphi+\frac{1}{2}\left(1-\frac{l}{f}\right)\mathrm{d}\tau, \quad H=\frac{1}{f}, \quad K_m=0. \tag{1.2.33}$$

The R-charges $+1$ and -1 are assigned to ξ and $\bar{\xi}$, respectively. And they are normalized to satisfy $\bar{\xi}\xi=-1$. Then the Killing vector field $v=v^m\partial_m$ with $v^m=\bar{\xi}\gamma^a\xi$ is as follows.

$$\bar{\xi}\gamma^a\xi=(0,-\sin\theta,-\cos\theta), \qquad v=-\frac{1}{\tilde{l}}\partial_\varphi-\frac{1}{l}\partial_\tau. \tag{1.2.34}$$

Note that another pair of Killing spinors, which is referred to as $\xi',\bar{\xi}'$ in Appendix A exists. They are associated with another set of background fields $\tilde{V}_m,\tilde{H},\tilde{K}_m$:

$$\tilde{V}=-\frac{1}{2}\left(1-\frac{\tilde{l}}{f}\right)\mathrm{d}\varphi+\frac{1}{2}\left(1-\frac{l}{f}\right)\mathrm{d}\tau, \quad \tilde{H}=-\frac{1}{f}, \quad \tilde{K}_m=0. \tag{1.2.35}$$

Therefore, once one chooses a set of background fields, the pair of Killing spinors associated with another set of background fields dose not correspond to SUSY of that background. Since $V_m=\tilde{V}_m=0$, $H=\tilde{H}=1/\ell$ on S^3, it is easy to see that both pairs do generate supersymmetry before the ellipsoidal deformation. In what follows, we take the background fields (1.2.33) and use $\xi,\bar{\xi}$ as our Killing spinors unless otherwise noted.

1.3 Supersymmetric transformation rules and Lagrangians

Given a supersymmetric background, one can derive the curved-space supersymmetric variation and Lagrangians by modifying the previous law(1.1.13), (1.1.15). First, one redefines the covariant derivative:

$$D_m \equiv (\nabla_m - r_\Phi V_m - iA_m)\,, \tag{1.3.1}$$

with R-charge $r_\Phi = R[\Phi]$, that is, $r_\Phi = (r, r-1, r-2)$ for the chiral multiplet $\Phi = (\phi, \psi, F)$ and ∇_m is defined in Section 1.2.2. For example, for the chiral spinor ψ and the anti-chiral spinor $\bar{\psi}$,

$$\begin{aligned} D_m\psi &= \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} - (r-1)V_m - iA_m\right)\psi, \\ D_m\bar{\psi} &= \left(\partial_m + \frac{1}{4}\Omega_m^{ab}\gamma^{ab} + (r-1)V_m\right)\bar{\psi} + i\bar{\psi}A_m \end{aligned} \tag{1.3.2}$$

Then, the supersymmetric transformation for chiral multiplet (1.1.15) is modified to

$$\begin{aligned} \boldsymbol{Q}\phi &= \xi\psi, \\ \boldsymbol{Q}\bar{\phi} &= \bar{\xi}\bar{\psi}, \\ \boldsymbol{Q}\psi &= i\left(\not{D}\phi + \sigma\phi\right)\bar{\xi} - r\phi\bar{\kappa} + F\xi, \\ \boldsymbol{Q}\bar{\psi} &= i\left(\not{D}\bar{\phi} + \bar{\phi}\sigma\right)\xi - r\bar{\phi}\kappa + \bar{F}\bar{\xi}, \\ \boldsymbol{Q}F &= i\bar{\xi}\left(\not{D}\psi - \sigma\psi\right) - i\bar{\xi}\bar{\lambda}\phi + \left(r - \frac{1}{2}\right)\bar{\kappa}\psi, \\ \boldsymbol{Q}\bar{F} &= i\xi\left(\not{D}\bar{\psi} - \bar{\psi}\sigma\right) + i\xi\bar{\phi}\lambda + \left(r - \frac{1}{2}\right)\kappa\bar{\psi}. \end{aligned} \tag{1.3.3}$$

The square of $\boldsymbol{Q}$ acts on the chiral multiplet as

$$\begin{aligned} \boldsymbol{Q}^2\phi &= iv^m\partial_m\phi + \Sigma\phi + rB\phi \\ \boldsymbol{Q}^2\bar{\phi} &= iv^m\partial_m\bar{\phi} - \bar{\phi}\Sigma - rB\bar{\phi} \\ \boldsymbol{Q}^2\psi &= iv^m\partial_m\psi + \Sigma\psi + (r-1)B\psi + \frac{1}{4}\left(v^m\Omega_m^{ab} + \Theta_{(v)}^{ab}\right)\gamma^{ab}\psi \\ \boldsymbol{Q}^2\bar{\psi} &= iv^m\partial_m\bar{\psi} - \bar{\psi}\Sigma - (r-1)B\bar{\psi} + \frac{1}{4}\left(v^m\Omega_m^{ab} + \Theta_{(v)}^{ab}\right)\gamma^{ab}\bar{\psi} \\ \boldsymbol{Q}^2F &= iv^m\partial_m F + \Sigma F + (r-2)BF \\ \boldsymbol{Q}^2\bar{F} &= iv^m\partial_m\bar{F} - \bar{F}\Sigma - (r-2)B\bar{F}. \end{aligned} \tag{1.3.4}$$

where

$$\begin{aligned} v^m &= \bar{\xi}\gamma^m\xi, & \Sigma &= v^m A_m - i\bar{\xi}\xi\cdot\sigma, \\ B &= \left(v^m V_m + \bar{\xi}\xi\cdot H\right), & \Theta_{(v)}^{ab} &= \nabla_{[m}v_{n]}e^{am}e^{bn}. \end{aligned} \tag{1.3.5}$$

Note that in order to derive the above result for $\boldsymbol{Q}^2F$, one needs the formula

$$\bar{\xi}\left(\not{D}\not{D}\phi\cdot\bar{\xi} + \frac{i}{2}F_{mn}\phi\gamma^{mn}\bar{\xi} + \frac{2r}{3}\phi\not{D}\not{D}\bar{\xi}\right) = 0. \tag{1.3.6}$$

This is automatically satisfied if ϕ and $\bar{\xi}$ couple to V_m according to their R-charge, namely the commutators of the covariant derivative act on those fields as follows.

$$\begin{aligned}
[\,D_m\,,D_m\,]\;\phi &= \,-iF_{mn}\phi - ir(\partial_m V_n - \partial_n V_m)\phi, \\
[\,D_m\,,D_n\,]\;\bar{\xi} &= \,\frac{1}{4}R^{ab}_{mn}\gamma^{ab}\bar{\xi} + i(\partial_m V_n - \partial_n V_m)\bar{\xi}.
\end{aligned} \tag{1.3.7}$$

Likewise, the supersymmetric transformation for vectormultiplet is

$$\begin{aligned}
\boldsymbol{Q}A_m &= -\frac{i}{2}\left(\bar{\xi}\gamma_m\lambda + \xi\gamma_m\bar{\lambda}\right), \\
\boldsymbol{Q}\sigma &= \frac{1}{2}\left(\xi\bar{\lambda} - \bar{\xi}\lambda\right), \\
\boldsymbol{Q}\lambda &= \frac{1}{2}\gamma^{mn}\xi F_{mn} - \xi D - i\not{D}\sigma\cdot\xi + \sigma\kappa, \\
\boldsymbol{Q}\bar{\lambda} &= \frac{1}{2}\gamma^{mn}\bar{\xi}F_{mn} + \bar{\xi}D + i\not{D}\sigma\cdot\bar{\xi} - \sigma\bar{\kappa}, \\
\boldsymbol{Q}D &= \frac{1}{2}\left(\xi\not{D}\bar{\lambda} - \xi\not{D}\bar{\lambda}\right) + \frac{i}{2}\left(\xi[\sigma,\bar{\lambda}] + \bar{\xi}[\sigma,\lambda]\right) + \frac{1}{4}(\kappa\bar{\lambda} - \bar{\kappa}\lambda).
\end{aligned} \tag{1.3.8}$$

The square of $\boldsymbol{Q}$ acts on them as follows.

$$\begin{aligned}
\boldsymbol{Q}^2A_m &= iv^n\partial_n A_m + i\partial_m v^n\cdot A_n - iD_m\Sigma, \\
\boldsymbol{Q}^2\sigma &= iv^m\partial_m\sigma + [\Sigma,\sigma], \\
\boldsymbol{Q}^2\lambda &= iv^m\partial_m\lambda + [\Sigma,\lambda] + B\lambda + \frac{1}{4}\left(v^m\Omega^{ab}_m + \Theta^{ab}_{(v)}\right)\gamma^{ab}\lambda, \\
\boldsymbol{Q}^2\bar{\lambda} &= iv^m\partial_m\lambda + [\Sigma,\bar{\lambda}] - B\bar{\lambda} + \frac{1}{4}\left(v^m\Omega^{ab}_m + \Theta^{ab}_{(v)}\right)\gamma^{ab}\bar{\lambda}, \\
\boldsymbol{Q}^2D &= iv^m\partial_m D + [\Sigma,D]
\end{aligned} \tag{1.3.9}$$

$\boldsymbol{Q}^2D$ takes the above form thanks to

$$\xi\not{D}\bar{\kappa} + \bar{\xi}\not{D}\kappa = 0, \tag{1.3.10}$$

which can be shown using only (1.2.28). The square of $\boldsymbol{Q}$, which is a sum of bosonic symmetry as noted, is modified to

$$\boldsymbol{Q}^2 = i\pounds_v + \mathsf{Gauge}_\Sigma + B\mathcal{R}_{U(1)}, \tag{1.3.11}$$

where the Lie derivatives for the fields are defined by (1.2.26).

Some $\boldsymbol{Q}$-invariant quantities, namely candidates of Lagrangian, are listed below. It is easy to check the $\boldsymbol{Q}$-invariance for $\mathcal{L}_{\rm CS}$ and $\mathcal{L}_{\rm FI}$. Those for $\mathcal{L}_{\rm YM}$ and $\mathcal{L}_{\rm mat}$ are shown in next Subsection 1.3.1.

$$\mathcal{L}_{\rm CS} = \frac{ik}{4\pi}\mathrm{Tr}\left[\varepsilon^{mnp}(A_m\partial_n A_p - \frac{2i}{3}A_m A_n A_p) - \bar{\lambda}\lambda - 2\sigma D\right], \tag{1.3.12}$$

$$\mathcal{L}_{\rm FI} = \frac{i\zeta}{2\pi}\big(D + H\sigma - K^m A_m\big), \tag{1.3.13}$$

$$\begin{aligned}\mathcal{L}_{\rm YM} &= \frac{1}{g^2}\mathrm{Tr}\bigg[\frac{1}{2}(F_{mn} - \varepsilon_{mnp}\sigma K^p)^2 + (D_m\sigma)^2 + \big(D - H\sigma\big)^2 \\ &\quad + \frac{i}{2}\bar{\lambda}\gamma^m D_m\lambda - \frac{i}{2}D_m\bar{\lambda}\gamma^m\lambda - i\bar{\lambda}[\sigma,\lambda] - \frac{1}{2}\bar{\lambda}(H + i\not{K})\lambda\bigg],\end{aligned} \tag{1.3.14}$$

$$\begin{aligned}\mathcal{L}_{\rm mat} &= D_m\bar{\phi}D^m\phi + \bar{\phi}\sigma^2\phi + i(2r-1)H\bar{\phi}\sigma\phi + \frac{r}{4}R\bar{\phi}\phi - i\bar{\phi}D\phi + \bar{F}F \\ &\quad - \frac{r(2r-1)}{2}(H^2 - K_m K^m)\bar{\phi}\phi + \frac{2r-1}{2}K^m(\bar{\phi}D_m\phi - D_m\bar{\phi}\phi) \\ &\quad - \frac{i}{2}\bar{\psi}\gamma^m D_m\psi + \frac{i}{2}D_m\bar{\psi}\gamma^m\psi - \frac{2r-1}{2}\bar{\psi}(H - i\not{K})\psi \\ &\quad + i\bar{\psi}\sigma\psi + i\bar{\psi}\bar{\lambda}\phi - i\bar{\phi}\lambda\psi.\end{aligned} \tag{1.3.15}$$

1.3.1 Exactness of $\mathcal{L}_{\rm YM}$ and $\mathcal{L}_{\rm mat}$

Here we would like to show the SUSY exactness of $\mathcal{L}_{\rm YM}, \mathcal{L}_{\rm mat}$ by finding $\mathcal{F}$ which satisfies $\boldsymbol{Q}\mathcal{F} = \mathcal{L}$ up to total derivatives.

Consider first the F-term of gauge-invariant chiral multiplet with R-charge $r = 2$,

$$\begin{aligned}&\boldsymbol{Q}\phi = \xi\psi, \qquad \boldsymbol{Q}\psi = i\not{D}\phi\bar{\xi} - 2\phi\bar{\kappa} + F\xi, \qquad \boldsymbol{Q}F = iD_m(\bar{\xi}\gamma^m\psi), \\ &\boldsymbol{Q}\bar{\phi} = \bar{\xi}\bar{\psi}, \qquad \boldsymbol{Q}\bar{\psi} = i\not{D}\bar{\phi}\xi - 2\bar{\phi}\kappa + \bar{F}\bar{\xi}, \qquad \boldsymbol{Q}\bar{F} = iD_m(\xi\gamma^m\bar{\psi}).\end{aligned} \tag{1.3.16}$$

One can show that the gauge-invariant F-term is $\boldsymbol{Q}$-exact up to total derivatives,

$$\boldsymbol{Q}(\bar{\eta}\psi) = F + D_m(i\bar{\eta}\gamma^m\bar{\xi}\phi), \qquad \boldsymbol{Q}(\eta\bar{\psi}) = \bar{F} + D_m(i\eta\gamma^m\xi\bar{\phi}) \tag{1.3.17}$$

if there are spinor fields $\eta, \bar{\eta}$ of R-charge $+1, -1$ satisfying

$$\bar{\eta}\xi = 1, \qquad \eta\bar{\xi} = 1, \tag{1.3.18}$$

and

$$\bar{\xi}\not{D}\bar{\eta} + \frac{i}{2}\bar{\kappa}\bar{\eta} = 0, \qquad \xi\not{D}\eta + \frac{i}{2}\kappa\eta = 0. \tag{1.3.19}$$

Note that the above equations imply that $\eta, \bar{\eta}$ are $\boldsymbol{Q}^2$-invariant:

$$\begin{aligned}\boldsymbol{Q}^2\eta &= i\pounds_v\eta + a\eta + \frac{1}{2}\omega\eta = -i\gamma^m\xi\cdot D_m(\eta\bar{\xi}) + i\bar{\xi}\cdot\left(\xi\not{D}\eta + \frac{i}{2}\kappa\eta\right), \\ \boldsymbol{Q}^2\bar{\eta} &= i\pounds_v\bar{\eta} - a\bar{\eta} + \frac{1}{2}\omega\bar{\eta} = -i\gamma^m\bar{\xi}\cdot D_m(\bar{\eta}\xi) + i\xi\cdot\left(\bar{\xi}\not{D}\bar{\eta} + \frac{i}{2}\bar{\kappa}\bar{\eta}\right).\end{aligned} \tag{1.3.20}$$

For the moment, we are unable to prove that every 3D background with Killing spinors $\xi, \bar{\xi}$ has spinor fields $\eta, \bar{\eta}$ satisfying the above property. So we present the explicit forms of $\eta, \bar{\eta}$ in interesting cases.

■ $\mathbb{R}^2 \times \mathbb{R}$, S^3_b Since we chose the Killing spinor (1.2.32) and (A.1.5) for S^3_b and $\mathbb{R}^2 \times \mathbb{R}$, respectively, so that $\bar{\xi}\xi = -1$, one finds that

$$\eta = \xi, \qquad \bar{\eta} = -\bar{\xi}, \tag{1.3.21}$$

satisfy $\eta\bar{\xi} = \bar{\eta}\xi = 1$. As $\kappa = \bar{\kappa} = 0$ on $\mathbb{R}^2 \times \mathbb{R}$, it is obvious that (1.3.19) is satisfied. While on S^3_b, since the background fields are $K_m = 0, H = \frac{1}{f}$, and $\kappa, \bar{\kappa}$ are proportional to $\xi, \bar{\xi}$, respectively. Thus, each term in (1.3.19) is zero due to $\xi\xi = \bar{\xi}\bar{\xi} = 0$.

■ $S^2 \times S^1$ Since we chose the Killing spinor (A.3.13) so that $\bar{\xi}\xi = -\cos\theta$, the situation is slightly different from the above example. However, the fact that the third component of the Killing vector $v^3 = \bar{\xi}\gamma^3\xi = -1$ indicates that

$$\eta = \gamma^3\xi, \qquad \bar{\eta} = \gamma^3\bar{\xi} \tag{1.3.22}$$

satisfy $\eta\bar{\xi} = \bar{\eta}\xi = 1$. Since the background fields are $K_m = \frac{1}{\ell}\delta_{m3}, H = 0$, we can write $\kappa, \bar{\kappa}$:

$$\kappa = -\frac{i}{\ell}\gamma^3\xi, \qquad \bar{\kappa} = \frac{i}{\ell}\gamma^3\bar{\xi}. \tag{1.3.23}$$

Thus, each term in (1.3.19) is zero as well.

Assuming that $\eta, \bar{\eta}$ satisfying (1.3.18),(1.3.19) exist, let us use the above argument to show the exactness of $\mathcal{L}_{\text{YM}}$. The field $\Phi_{\text{YM}} \equiv \frac{1}{2}\operatorname{Tr}\lambda^2$ is the bottom component of a gauge-invariant chiral multiplet with $r = 2$. If we define the higher component Ψ_{YM} by

$$\boldsymbol{Q}\Phi_{\text{YM}} = \xi\Psi_{\text{YM}}, \tag{1.3.24}$$

its explicit form reads

$$\Psi_{\text{YM}} = \operatorname{Tr}\left[\left\{-\frac{1}{2}\gamma^{mn}f_{mn} - D + i\not{D}\sigma + \sigma(H + i\not{K})\right\}\lambda\right]. \tag{1.3.25}$$

As in (1.3.17), F-term is thus calculated as follows.

$$\begin{aligned}\boldsymbol{Q}(\bar{\eta}\Psi_{\text{YM}}) = \operatorname{Tr}\Big[&(D - \sigma H)^2 + (\tilde{F}_m - \sigma K_m - D_m\sigma)^2 \\ &- iD_m\bar{\lambda}\gamma^m\lambda - i\bar{\lambda}[\sigma, \lambda] - \frac{1}{2}\bar{\lambda}(H + i\not{K})\lambda\Big] + D_m(i\bar{\eta}\gamma^m\bar{\xi}\Phi_{\text{YM}}),\end{aligned} \tag{1.3.26}$$

where we used the notation $\tilde{F}_m = \frac{1}{2}\varepsilon_{mnl}F^{nl}$. Likewise, starting from an anti-chiral field $\bar{\Phi}_{\text{YM}} = \frac{1}{2}\operatorname{Tr}\bar{\lambda}^2$, one can derive

$$\begin{aligned}\boldsymbol{Q}\bar{\Phi}_{\text{YM}} &= \bar{\xi}\bar{\Psi}_{\text{YM}}, \\ \bar{\Psi}_{\text{YM}} &= \operatorname{Tr}\left[\left\{-\frac{1}{2}\gamma^{mn}F_{mn} + D - i\not{D}\sigma - \sigma(H - i\not{K})\right\}\bar{\lambda}\right], \\ \boldsymbol{Q}(\eta\bar{\Psi}_{\text{YM}}) &= \operatorname{Tr}\Big[(D - \sigma H)^2 + (\tilde{F}_m - \sigma K_m + D_m\sigma)^2 \\ &\qquad + i\lambda\bar{\not{D}}\lambda - i\bar{\lambda}[\sigma, \lambda] - \frac{1}{2}\bar{\lambda}(H + i\not{K})\lambda\Big] + D_m(i\eta\gamma^m\xi\bar{\Phi}_{\text{YM}}).\end{aligned} \tag{1.3.27}$$

The sum of (1.3.25) and (1.3.27) is exactly the Yang-Mills Lagrangian (1.3.14) up to total derivatives,

$$\mathcal{L}_{\mathrm{YM}} = \frac{1}{2g^2}\boldsymbol{Q}\big(\bar{\eta}\Psi_{\mathrm{YM}} + \eta\Psi_{\mathrm{YM}}\big). \tag{1.3.28}$$

Next we turn to show the exactness of $\mathcal{L}_{\mathrm{mat}}$. We take $\Phi_{\mathrm{mat}} = \bar{F}\phi$ as the lowest component of a gauge-invariant chiral multiplet of R-charge 2.

$$\begin{aligned}
\boldsymbol{Q}\Phi_{\mathrm{mat}} =& \xi\Psi_{\mathrm{mat}},\\
\Psi_{\mathrm{mat}} =& \bar{F}\psi + i\not{D}\bar{\psi}\phi + i\bar{\psi}\sigma\phi + i\bar{\phi}\lambda\phi + \big(r-\frac{1}{2}\big)(H+i\not{K})\bar{\psi}\phi,\\
\boldsymbol{Q}(\bar{\eta}\Psi_{\mathrm{mat}}) =& -D_mD^m\bar{\phi}\phi + \bar{\phi}\sigma^2\phi + i(2r-1)H\bar{\phi}\sigma\phi + \frac{r}{4}R\bar{\phi}\phi - i\bar{\phi}D\phi + \bar{F}F\\
&- \frac{r(2r-1)}{2}(H^2 - K_mK^m)\bar{\phi}\phi - (2r-1)K^mD_m\bar{\phi}\phi\\
&+ iD_m\bar{\psi}\gamma^m\psi - \frac{2r-1}{2}\bar{\psi}(H-i\not{K})\psi\\
&+ i\bar{\psi}\sigma\psi + i\bar{\psi}\bar{\lambda}\phi - i\bar{\phi}\lambda\psi + D_m(i\bar{\xi}\gamma^m\bar{\eta}\Phi_{\mathrm{mat}})
\end{aligned} \tag{1.3.29}$$

Likewise, taking anti-chiral field $\bar{\Phi}_{\mathrm{mat}} = \bar{\phi}F$, one can derive

$$\begin{aligned}
\boldsymbol{Q}\bar{\Phi}_{\mathrm{mat}} =& \bar{\xi}\bar{\Psi}_{\mathrm{mat}},\\
\Psi_{\mathrm{mat}} =& \bar{\psi}F + i\bar{\phi}\not{D}\psi - i\bar{\phi}\sigma\psi - i\bar{\phi}\bar{\lambda}\phi + \big(r-\frac{1}{2}\big)(H-i\not{K})\bar{\phi}\psi,\\
\boldsymbol{Q}(\eta\bar{\Psi}_{\mathrm{mat}}) =& -\bar{\phi}D_mD^m\phi + \bar{\phi}\sigma^2\phi + i(2r-1)H\bar{\phi}\sigma\phi + \frac{r}{4}R\bar{\phi}\phi - i\bar{\phi}D\phi + \bar{F}F\\
&- \frac{r(2r-1)}{2}(H^2 - K_mK^m)\bar{\phi}\phi - (2r-1)K^mD_m\bar{\phi}\phi\\
&- i\bar{\psi}\gamma^mD_m\psi - \frac{2r-1}{2}\bar{\psi}(H-i\not{K})\psi\\
&+ i\bar{\psi}\sigma\psi + i\bar{\psi}\bar{\lambda}\phi - i\bar{\phi}\lambda\psi + D_m(i\bar{\xi}\gamma^m\bar{\eta}\Phi_{\mathrm{mat}}).
\end{aligned} \tag{1.3.30}$$

Thus, the sum of these F-terms is $\mathcal{L}_{\mathrm{mat}}$ up to total derivatives,

$$\mathcal{L}_{\mathrm{mat}} = \frac{1}{2}\boldsymbol{Q}\big(\bar{\eta}\Psi_{\mathrm{mat}} + \eta\Psi_{\mathrm{mat}}\big). \tag{1.3.31}$$

In this section, we showed the $\boldsymbol{Q}$-exactness of $\mathcal{L}_{\mathrm{YM}}$ and $\mathcal{L}_{\mathrm{mat}}$. Throughout the calculations, we kept track of total derivative terms even though they were dropped in the final results (1.3.28),(1.3.31). They will become important in Section 1.5.

1.4 Vortex operators

Vortex operators are one-dimensional defects in 3D gauge theories characterized by a singular behavior of the gauge field. For simplicity, let us suppose that there is a vortex line along the x^3-axis of $\mathbb{R}^3$. It is the simplest to describe it using cylindrical coordinate, i.e. r, φ are the polar coordinates for the transverse (x_1, x_2)-plane, and the x_3-axis is the cylindrical axis t as in Appendix A.1. We require that the gauge field behaves near it as

$$A \sim \beta\, \mathrm{d}\varphi. \tag{1.4.1}$$

If the gauge field A precisely takes the above value, the gauge field strength is $F_{12} = 2\pi\beta\delta^2(x_1, x_2)$. The 1-form $\mathrm{d}\varphi = (x_1 \mathrm{d}x_2 - x_2 \mathrm{d}x_1)/((x_1)^2 + (x_2)^2)$ gets larger and larger as it approaches $x_1 = x_2 = 0$ and finally diverges at that point. Hence, the requirement (1.4.1) introduces a defect operator in the shape of a vortex along t. The parameter β, the coefficient of $\mathrm{d}\varphi$, is called the *vorticity* as it represents its magnitude. β is a constant that takes values in $\mathfrak{g} = \mathrm{Lie}(G)$, but it can be gauge-rotated to be in a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Upon the vortex line the gauge group G is broken to a subgroup K of G which is the centralizer of β, namely the group of elements of G which commute with β

$$K = \left\{ h \in G | h\beta h^{-1} = \beta \right\}. \tag{1.4.2}$$

K is $U(1)^r$ ($r = \mathrm{rank}(G)$) for a generic β. If one takes a special β, it can be non-abelian.

What are the conditions for the supersymmetry to be preserved after inserting the above vortex operator? Since the square of $\boldsymbol{Q}$ involves a Lie derivative in the direction of the Killing vector $v = \bar{\xi}\gamma^a \xi \partial_a$, for a vortex operator to be supersymmetric, it must extend along v. Now, in the case $v = -\partial_t = -\partial_3$, $\boldsymbol{Q}^2(\text{fields}) = 0$ implies

$$D_3\sigma = D_3 D = F_{3a} = 0. \tag{1.4.3}$$

Also $\xi, \bar{\xi}$ must be eigenspinors of γ^3

$$\gamma^3 \xi = +\xi, \qquad \gamma^3 \bar{\xi} = -\bar{\xi}, \tag{1.4.4}$$

which follow from the identities:

$$v^a \gamma^a \xi = -\xi, \qquad v^a \gamma^a \bar{\xi} = +\bar{\xi}. \tag{1.4.5}$$

Indeed, the Killing spinors we chose (A.1.5) satisfy the above equations. The SUSY transformations $\boldsymbol{Q}$ of A_m, σ, D are trivial as the classical configuration of gaugino $\lambda, \bar{\lambda}$ is zero. Then, non-trivial equations are as follows:

$$\begin{aligned} 0 &= \boldsymbol{Q}\lambda = \frac{1}{2}\gamma^{ab}\xi F_{ab} - \xi D - i\not{D}\sigma \cdot \xi, \\ 0 &= \boldsymbol{Q}\bar{\lambda} = \frac{1}{2}\gamma^{ab}\bar{\xi} F_{ab} + \bar{\xi} D + i\not{D}\sigma \cdot \bar{\xi}, \end{aligned} \tag{1.4.6}$$

which are rewritten in the following form

$$\begin{aligned} 0 &= \begin{pmatrix} iF_{12} - D & -i(D_1\sigma - iD_2\sigma) \\ i(D_1\sigma + iD_2\sigma) & -iF_{12} - D \end{pmatrix} \begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix}, \\ 0 &= \begin{pmatrix} iF_{12} + D & i(D_1\sigma - iD_2\sigma) \\ -i(D_1\sigma + iD_2\sigma) & -iF_{12} + D \end{pmatrix} \begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix}. \end{aligned} \tag{1.4.7}$$

Therefore one obtains, in addition to (1.4.3), the following conditions.

$$D = iF_{12}, \quad D_1\sigma = D_2\sigma = 0. \tag{1.4.8}$$

If these BPS conditions are satisfied, the SUSY (1.4.4) is preserved even after inserting the vortex operator [21, 22, 35]. Especially, when only two of four SUSY are preserved as in the present case[9], it is called a half-BPS. Note that this condition is compatible with the saddle point condition, which will be described in detail later. This fact is helpful for later calculation using localization techniques.

1.5 SUSY with boundary

Naive volume integral of Lagrangians may be divergent in the presence of a vortex line. As in the previous section, we assume there is a single vortex line along x_3-axis, and we use the standard cylindrical coordinate (r, φ, t). The flat metric and vielbein on the $\mathbb{R}^3$ are expressed as follows.

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\varphi^2 + \mathrm{d}t^2, \qquad e^1 = \mathrm{d}r, \quad e^2 = r\mathrm{d}\varphi, \quad e^3 = \mathrm{d}t. \tag{1.5.1}$$

As in [9], we regularize the volume integral by removing a tubular neighborhood of the line $r \le \epsilon$ from the integration domain and adding appropriate boundary terms at $r = \epsilon$, so that the sum of bulk and boundary terms

$$S + S_{\mathrm{B}} = \int_{r\ge\epsilon} \mathrm{d}V\mathcal{L} + \int_{r=\epsilon} \mathrm{d}S\mathcal{L}_{\mathrm{B}} \qquad \big(\mathrm{d}V \equiv e^1e^2e^3, \quad \mathrm{d}S \equiv e^2e^3\big)$$

is SUSY invariant. On the boundary, all fields of the theory must be provided with some boundary conditions. This will be discussed in Section 2.3. Here we suppose that, in addition to $A_\varphi|_{r=\epsilon} = \beta$, some boundary conditions are given for all other fields.

For some of the Lagrangians, the boundary terms can be found by using the argument given in Section 1.3.1: the F-component of a gauge-invariant chiral multiplet (Φ, Ψ, F_Φ) with $r = 2$ is $\boldsymbol{Q}$-exact up to a total derivative. More explicitly, the following holds:

$$\boldsymbol{Q}(\bar\eta\Psi) = F_\Phi + D_m(i\bar\eta\gamma^m\bar\xi\Phi), \qquad \boldsymbol{Q}(\eta\bar\Psi) = \bar F_\Phi + D_m(i\eta\gamma^m\xi\bar\Phi). \tag{1.5.2}$$

As an example, $\mathcal{L}_{\mathrm{YM}}$ could be expressed as $F_\Phi + \bar F_\Phi$ for a gauge invariant chiral field $\Phi_{\mathrm{YM}} = \frac{1}{2g^2}\mathrm{Tr}\lambda\lambda$ and its conjugate $\bar\Phi_{\mathrm{YM}} = \frac{1}{2g^2}\mathrm{Tr}\bar\lambda\bar\lambda$. The exactness of $\mathcal{L}_{\mathrm{YM}}$ (1.3.28) is rewritten including total derivatives as follows.

$$\frac{1}{2g^2}\big(\boldsymbol{Q}(\bar\eta\Psi_{\mathrm{YM}}) + \boldsymbol{Q}(\eta\Psi_{\mathrm{YM}})\big) = \mathcal{L}_{\mathrm{YM}} + \frac{i}{2g^2}D_m\Big(\bar\eta\gamma^m\bar\xi\ \mathrm{Tr}\,\lambda\lambda + \eta\gamma^m\xi\ \mathrm{Tr}\,\bar\lambda\bar\lambda\Big) \tag{1.5.3}$$

The boundary term for $\mathcal{L}_{\mathrm{YM}}$ is thus given by

$$\mathcal{L}_{\mathrm{YM,B}} = \frac{i}{2g^2}\Big(\bar\eta\gamma^1\bar\xi\,\mathrm{Tr}\lambda\lambda + \eta\gamma^1\xi\,\mathrm{Tr}\bar\lambda\bar\lambda\Big). \tag{1.5.4}$$

Similarly, by setting $\Phi_{\mathrm{mat}} = \frac{1}{2}\bar F\phi, \bar\Phi_{\mathrm{mat}} = \frac{1}{2}\bar\phi F$ one obtains $\mathcal{L}_{\mathrm{mat}}$ as their F-components up to total derivatives. This allows us to determine the boundary term for $\mathcal{L}_{\mathrm{mat}}$ as follows:

$$\mathcal{L}_{\mathrm{mat,B}} = \frac{1}{2}\Big(i\bar\eta\gamma^1\bar\xi\cdot\bar F\phi + i\eta\gamma^1\xi\cdot\bar\phi F - D^1(\bar\phi\phi)\Big). \tag{1.5.5}$$

[9] $\xi, \bar\xi$ satisfy Eq. (1.4.4), but $\xi', \bar\xi'$ do not. So ξ and $\bar\xi$ are associated to the preserved SUSY.

The boundary terms for $\mathcal{L}_{\rm FI}$ and $\mathcal{L}_{\rm CS}$ can be constructed using the following argument. Generally, supersymmetric bulk Lagrangian $\mathcal{L}$ satisfies $\boldsymbol{Q}\mathcal{L} = D_m\mathcal{V}^m$ for some $\mathcal{V}^m$. If $\mathcal{V}^1$ is $\boldsymbol{Q}$-exact, the boundary term can be determined from $\mathcal{V}^1 = -\boldsymbol{Q}\mathcal{L}_{\rm B}$. By applying this to $\mathcal{L}_{\rm FI}$, first we find

$$\mathcal{V}^m_{\rm FI} = \frac{\zeta}{4\pi}\left(\bar{\xi}\gamma^m\lambda - \xi\gamma^m\bar{\lambda}\right).$$

After some manipulations we can write $\mathcal{V}^1_{\rm FI}$ in a $\boldsymbol{Q}$-exact form:

$$\begin{aligned}
\mathcal{V}^1_{\rm FI} &= \frac{\zeta}{4\pi}(\bar{\xi}\gamma^1\lambda - \xi\gamma^1\bar{\lambda}) \\
&= -\frac{\zeta}{4\pi}v_m(\bar{\xi}\gamma^m\gamma^1\lambda + \xi\gamma^m\gamma^1\bar{\lambda}) \\
&= \frac{i\zeta}{4\pi}\varepsilon^{1mn}v_m(\bar{\xi}\gamma_n\lambda + \xi\gamma_n\bar{\lambda}) \\
&= -\frac{\zeta}{2\pi}\boldsymbol{Q}(w^nA_n), \qquad w^n \equiv \varepsilon^{1mn}v_m.
\end{aligned} \tag{1.5.6}$$

Here we used $\not{v}\xi = -\xi$, $\not{v}\bar{\xi} = \bar{\xi}$ at the second equality and $v^1 = 0$ at the third equality. Similar analysis can be performed also for $\mathcal{L}_{\rm CS}$. Acting $\boldsymbol{Q}$ on $\mathcal{L}_{\rm CS}$, one finds

$$\mathcal{V}_{\rm CS}{}^m = \frac{ik}{4\pi}\,\mathrm{Tr}\left[\frac{1}{2}A_n(\bar{\xi}\gamma^{mn}\lambda + \xi\gamma^{mn}\bar{\lambda}) - i\sigma(\bar{\xi}\gamma^m\lambda - \xi\gamma^m\bar{\lambda})\right].$$

One thus finds the following boundary terms:

$$\mathcal{L}_{\rm FI,B} = \frac{\zeta}{2\pi}w^mA_m, \qquad \mathcal{L}_{\rm CS,B} = \frac{ik}{4\pi}\mathrm{Tr}\Big[w^mA_m(2i\sigma + v^nA_n)\Big]. \tag{1.5.7}$$

The derivation of both requires $v^1 = 0$, which means that the Killing vector v has to lie along the boundary in order for SUSY-preserving boundary terms to exist.

Chapter 2

Partition functions and vortex loop VEVs

If a theory has at least one supersymmetry realized off-shell, SUSY-invariant quantities defined by path integration can be evaluated by the *localization technique*. Localization principle allows one to reduce an infinite-dimensional SUSY path integral to a finite-dimensional integral over the configurations called saddle points. It was first applied to 3D SUSY gauge theories on S^3 in [1], then generalized to the theory with arbitrary R-charge assignment [3] and squashed S^3 [5].

In the first half of this chapter we will review this technique and derive the formulae for the partition function on ellipsoid, where we introduce a powerful prescription that was developed in [29]. This clarifies the computation process, and one finds that the set of eigenvalues of $\boldsymbol{Q}^2$ is the only things we need. It is a powerful method to compute not only the partition functions, but also the vacuum expectation value (VEV) of the vortex operator defined in Section 1.4. In the second half of this chapter, our goal is to derive the exact formulae for the VEV, and we will see that these results depend on the vorticity β as well as the choices of boundary conditions.

2.1 Path integration with localization technique

Supersymmetric path integrals localize to $\boldsymbol{Q}$-invariant field configurations or *saddle points*, so that the sum of Gaussian path-integrals (*one-loop determinants*) on each saddle point gives an exact answer. See [36]for a review of localization techniques in SUSY gauge theories. Saddle point configurations are the solutions of $\boldsymbol{Q}\Psi = 0$ for all the fermions Ψ of the theory.

First, let us consider the saddle point configurations without vortex loop operators. Taking $\Psi_{\rm YM}, \bar{\Psi}_{\rm YM}$ in (1.3.25),(1.3.27) as the fermion fields, the saddle point condition $\boldsymbol{Q}\Psi = 0$ becomes

$$0 = \mathrm{Tr}\left[\frac{1}{2}(F_{mn} - \varepsilon_{mnp}\sigma K^p)^2 + (D_m\sigma)^2 + (D - \sigma H)^2 + \cdots\right], \tag{2.1.1}$$

where the ellipses represent the terms including fermion fields that are usually set to zero on saddle points. Assuming suitable reality condition on bosonic fields, the values of the vectormultiplet fields at the saddle points are thus given by

$$F_{mn} = \varepsilon_{mnp}\sigma K^p, \qquad \sigma(\text{constant}), \qquad D = \sigma H. \tag{2.1.2}$$

Note that a $\mathrm{Lie}(G) = \mathfrak{g}$ valued constant σ can be gauge-rotated to take values in a Cartan subalgebra $\mathfrak{h}$. Likewise, the requirement that the $\boldsymbol{Q}$-variation of $\Psi_{\text{mat}}, \bar{\Psi}_{\text{mat}}$ in (1.3.29),(1.3.30) vanishes gives the saddle point configurations for chiral multiplets as follows.

$$\phi = F = 0, \qquad \bar{\phi} = \bar{F} = 0. \tag{2.1.3}$$

The conditions $\boldsymbol{Q}\Psi = 0$ for the other fermion fields are automatically satisfied by (2.1.2), (2.1.3). In the next section, we review an explicit computation of exact partition functions and VEVs of the vortex loop operator introduced in previous chapter.

2.2 The partition function on ellipsoid

On an ellipsoid preserving supersymmetry $\xi, \bar{\xi}$ (A.2.13), the background fields are given by

$$\begin{gathered} K_m = 0, \qquad H = \frac{1}{f}, \\ V = \frac{1}{2}\Big(1 - \frac{\tilde{\ell}}{f}\Big)d\phi + \frac{1}{2}\Big(1 - \frac{\ell}{f}\Big)d\tau. \end{gathered} \tag{2.2.1}$$

According to the argument in Section 2.1, supersymmetric path integrals localize to the saddle points:

$$A_m = 0, \qquad \sigma(\text{constant}) \in \mathfrak{h}, \qquad D = \frac{\sigma}{f}, \qquad \phi = F = 0. \tag{2.2.2}$$

In other words, the saddle points are labeled by constant values of σ. The FI (1.3.13) and CS (1.3.12) actions take the following classical values on these saddle points.

$$S_{\text{FI}} = 2\pi i \zeta \ell\tilde{\ell}\sigma, \qquad S_{\text{CS}} = -i\pi k \ell\tilde{\ell}\,\mathrm{Tr}\,\sigma^2 \tag{2.2.3}$$

The YM (1.3.14) and matter (1.3.15) actions vanish on the saddle points since they are $\boldsymbol{Q}$-exact. The other contributions to the path integral are one-loop determinants $\Delta_{\text{1-loop}}$ which are Gaussian integrals of the field fluctuations around these saddle points. In the following we present explicit calculations of that for both chiral and vectormultiplets.

2.2.1 One-loop determinants: chiral multiplet

One-loop determinants can be computed most easily by a suitable change of path-integration variables. Let us first explain this procedure for the theory of a chiral multiplet of unit $U(1)$ charge, with the $U(1)$ vectormultiplet fields fixed at a saddle point (2.2.2). The problem is already Gaussian, but it can be simplified further by rewriting in terms of the so-called *cohomological variables*

$$\Psi \equiv \boldsymbol{Q}\phi = \xi\psi, \quad \Psi' \equiv \bar{\eta}\psi, \quad F' \equiv \boldsymbol{Q}\Psi' = F + \mathcal{J}\phi\,; \quad \mathcal{J} \equiv i\bar{\eta}\gamma^m\bar{\xi}D_m\,. \tag{2.2.4}$$

The change of path integration variables from (ϕ, ψ, F) to (ϕ, Ψ, Ψ', F') is invertible and the Jacobian is trivial. ϕ is Grassmann-even and its superpartner Ψ is odd, and they are both scalars of R-charge r. Likewise, Ψ' (odd) and its superpartner F' (even) are both scalars of R-charge $r-2$. We denote the Hilbert spaces of their wavefunctions as

$$\phi, \Psi \in \mathcal{H}, \qquad \Psi', F' \in \mathcal{H}'.$$

Physically this means that the fields ϕ and Ψ are to be mode-expanded using the same set of basis wavefunctions of $\mathcal{H}$, and similarly for Ψ' and F' in $\mathcal{H}'$.

The one-loop determinant $\Delta_{\text{1-loop}}$ can be computed by path integrating over the fields (ϕ, Ψ, Ψ', F') and their conjugates with a suitable choice of localizing Lagrangian $\mathcal{L}$. Any $\mathcal{L}$ will do as long as it is $\boldsymbol{Q}$-exact and its bosonic part is bounded from below. Let us take[10]

$$\mathcal{L} = \boldsymbol{Q}\Big(\bar{\phi} \cdot \overline{\boldsymbol{Q}^2}\Psi + \bar{\Psi}' F'\Big) = \bar{\Psi} \cdot \overline{\boldsymbol{Q}^2}\Psi + \bar{\phi} \cdot \overline{\boldsymbol{Q}^2}\boldsymbol{Q}^2\phi + \bar{F}'F' - \bar{\Psi}'\boldsymbol{Q}^2\Psi',$$

Then the Gaussian integration gives the ratio of determinants

$$\Delta_{\text{1-loop}} \;=\; \frac{\text{Det}(\overline{\boldsymbol{Q}^2})_{\mathcal{H}} \cdot \text{Det}(-\boldsymbol{Q}^2)_{\mathcal{H}'}}{\text{Det}(\overline{\boldsymbol{Q}^2}\boldsymbol{Q}^2)_{\mathcal{H}}} \;=\; \frac{\text{Det}(\boldsymbol{Q}^2)_{\mathcal{H}'}}{\text{Det}(\boldsymbol{Q}^2)_{\mathcal{H}}}. \tag{2.2.5}$$

The last equality holds up to a sign factor $\text{Det}(-1)_{\mathcal{H}'}$ which we have just dropped. So $\Delta_{\text{1-loop}}$ can be computed from the spectrum of $\boldsymbol{Q}^2$ on $\mathcal{H}$ and $\mathcal{H}'$. Furthermore, one can check that the map $\mathcal{J} : \mathcal{H} \to \mathcal{H}'$ commutes with $\boldsymbol{Q}^2$, which is as expected because it is made only of $\boldsymbol{Q}^2$-invariant background fields. The $\boldsymbol{Q}^2$-eigenmodes in $\mathcal{H}$ and $\mathcal{H}'$ paired by $\mathcal{J}$ make no net contribution to $\Delta_{\text{1-loop}}$. Hence we only need the spectrum of $\boldsymbol{Q}^2$ on the kernel and cokernel of $\mathcal{J}$. In other words,

$$\Delta_{\text{1-loop}} \;=\; \frac{\text{Det}(\boldsymbol{Q}^2)_{\text{coker}(\mathcal{J})}}{\text{Det}(\boldsymbol{Q}^2)_{\text{ker}(\mathcal{J})}}\,. \tag{2.2.6}$$

To work out the basis wavefunctions of $\text{ker}(\mathcal{J})$ and $\text{coker}(\mathcal{J}) = \text{ker}(\bar{\mathcal{J}})$, we need the explicit form of $\mathcal{J}$ and its conjugate $\bar{\mathcal{J}}$.

$$\begin{aligned}
\mathcal{J} &= -ie^{-i(\varphi+\tau)}\left[-\frac{1}{f}\partial_\theta + \frac{i\cos\theta}{\tilde{\ell}\sin\theta}(\partial_\varphi - irV_\varphi) - \frac{i\sin\theta}{\ell\cos\theta}(\partial_\tau - irV_\tau)\right], \\
\bar{\mathcal{J}} &= +ie^{+i(\varphi+\tau)}\left[-\frac{1}{f}\partial_\theta - \frac{i\cos\theta}{\tilde{\ell}\sin\theta}(\partial_\varphi - i(r-2)V_\varphi) + \frac{i\sin\theta}{\ell\cos\theta}(\partial_\tau - i(r-2)V_\tau)\right].
\end{aligned} \tag{2.2.7}$$

In fact $\bar{\mathcal{J}}$ can be expressed as $\bar{\mathcal{J}} = -i\eta\gamma^m\xi D_m$, where

$$\eta\gamma^a\xi = e^{i(\varphi+\tau)}(1, i\cos\theta, -i\sin\theta), \quad \bar{\eta}\gamma^a\bar{\xi} = e^{-i(\varphi+\tau)}(1, -i\cos\theta, -\sin\theta). \tag{2.2.8}$$

The zeromode equations $\mathcal{J}\Phi = 0, \bar{\mathcal{J}}\Phi' = 0$ can be reduced to ordinary differential equations (ODEs) for functions of θ by assuming that Φ, Φ' have definite φ and τ-momenta. The resulting

[10]One can check that $\boldsymbol{Q}^2$ and $\overline{\boldsymbol{Q}^2}$ commute. Note also that there is no issue of boundary terms for this $\mathcal{L}$ since $\boldsymbol{Q}^2$ and $\overline{\boldsymbol{Q}^2}$ contain no θ-derivatives.

ODEs actually need not be solved explicitly, but the behavior of the solutions at $\theta = 0$ and $\pi/2$ are important. They are summarized as follows.

$$\begin{aligned}
\Phi &= \hat{\Phi}(\theta)e^{im\varphi+in\tau} \quad \in \ker(\mathcal{J}) \quad \Longrightarrow \quad \hat{\Phi}(\theta) \sim (\sin\theta)^{-m}(\cos\theta)^{-n}, \\
\Phi' &= \hat{\Phi}'(\theta)e^{im'\varphi+in'\tau} \in \ker(\bar{\mathcal{J}}) \quad \Longrightarrow \quad \hat{\Phi}'(\theta) \sim (\sin\theta)^{m'}(\cos\theta)^{n'},
\end{aligned} \tag{2.2.9}$$

with integers m, n, m', n'. One should require $m, n \leq 0$ and $m', n' \geq 0$ so that the zeromodes are regular. On the ellipsoid and for scalar fields, $\boldsymbol{Q}^2$ acts as

$$\begin{aligned}
\boldsymbol{Q}^2 &= i\pounds_v + \mathsf{Gauge}_{(v^m A_m + i\sigma)} + (v^m V_m - H)\mathcal{R}_{U(1)} \\
&= -\frac{i}{\tilde{\ell}}\partial_\phi - \frac{i}{\ell}\partial_\tau + i\left(\sigma + \frac{i}{\tilde{\ell}}A_\phi + \frac{i}{l}A_\tau\right) - \frac{1}{2}\left(\frac{1}{\tilde{\ell}} + \frac{1}{\ell}\right)\mathcal{R}_{U(1)},
\end{aligned} \tag{2.2.10}$$

where $A_\phi = A_\tau = 0$ on the saddle point. By multiplying all the eigenvalues of (2.2.10) we obtain

$$\Delta_{\text{1-loop}} = \frac{\prod_{m'n'\geq 0} \frac{m'}{\tilde{\ell}} + \frac{n'}{\ell} + i\sigma - \frac{r-2}{2}\left(\frac{1}{\tilde{\ell}} + \frac{1}{\ell}\right)}{\prod_{m,n\leq 0} \frac{m}{\tilde{\ell}} + \frac{n}{\ell} + i\sigma - \frac{r}{2}\left(\frac{1}{\tilde{\ell}} + \frac{1}{\ell}\right)}. \tag{2.2.11}$$

Now we introduce the notations,

$$b \equiv (\ell/\tilde{\ell})^{\frac{1}{2}}, \qquad Q \equiv b + b^{-1}, \qquad \hat{\sigma} \equiv \sqrt{\ell\tilde{\ell}}\sigma. \tag{2.2.12}$$

where the parameter b is referred to as the *squashing parameter*, and $\hat{\sigma}$ has mass dimension zero. Thus the formula (2.2.11) can be expressed as follows:

$$\begin{aligned}
\Delta_{\text{1-loop}} &= \prod_{m,n\geq 0} \frac{mb + nb^{-1} + i\hat{\sigma} - \frac{Q}{2}(r-2)}{mb + nb^{-1} - i\hat{\sigma} + \frac{Q}{2}r} = s_b\left(\frac{i(r-2)Q}{2} - \hat{\sigma}\right), \\
s_b(x) &= \prod_{m,n\geq} \frac{mb + nb^{-1} + ix + \frac{Q}{2}}{mb + nb^{-1} - ix + \frac{Q}{2}}.
\end{aligned} \tag{2.2.13}$$

Here $s_b(x)$ is the double sine function which satisfies the following relations.

$$\begin{aligned}
s_b(x) &= s_{1/b}(x) = s_b(-x)^{-1}, \\
s_b(\frac{ib}{2} - x)s_b(\frac{ib}{2} + x) &= 2\cosh^{-1}(\pi bx), \\
\frac{s_b(x \pm ib)}{s_b(x)} &= i\left(2\sinh\pi b\left(x \pm \frac{iQ}{2}\right)\right)^{\mp 1}.
\end{aligned} \tag{2.2.14}$$

For more detail on this function, we refer to [37,38], [39, appendix A.2].

The above result can be easily generalized to the theory of chiral multiplet in a representation $\mathbf{R}$ of the gauge group G. The one-loop determinant of a chiral multiplet with R-charge r is given by a product over weights μ of $\mathbf{R}$.

$$\Delta^{\mathrm{c}}_{\text{1-loop}} = \prod_{\mu} s_b\left(\frac{i(1-r)Q}{2} - i\mu\cdot\hat{\sigma}\right). \tag{2.2.15}$$

2.2.2 One-loop determinants: vector multiplet

Let us next study the integration over fluctuations of vectormultiplet fields around a saddle point (2.2.2). In what follows we denote the saddle-point value of a field Φ by $\langle\Phi\rangle$ and its fluctuation by $\delta\Phi \equiv \Phi - \langle\Phi\rangle$. As in [29], we first introduce the Faddeev-Popov ghost c, antighost $\bar{c}$ and an auxiliary field B and then move to cohomological variables.

The system of physical fields and ghosts has a nilpotent BRST symmetry $\boldsymbol{Q}_{\mathrm{B}}$. It acts on all the physical fields as gauge transformation with parameter c:

$$\begin{aligned} \boldsymbol{Q}_{\mathrm{B}} A_m &= D_m c, \qquad & \boldsymbol{Q}_{\mathrm{B}} \sigma &= i[\, c\,, \sigma], \\ \boldsymbol{Q}_{\mathrm{B}} \phi &= i c \phi, \qquad & \boldsymbol{Q}_{\mathrm{B}} \bar{\phi} &= -i \bar{\phi} c, \end{aligned} \tag{2.2.16}$$

whereas the ghost fields transform as

$$\boldsymbol{Q}_{\mathrm{B}} c = i c^2, \quad \boldsymbol{Q}_{\mathrm{B}} \bar{c} = B, \quad \boldsymbol{Q}_{\mathrm{B}} B = 0. \tag{2.2.17}$$

It is also known from [40] that if we set

$$\boldsymbol{Q} c = i\delta\Sigma, \quad \boldsymbol{Q}\bar{c} = 0, \quad \boldsymbol{Q} B = i v^m \partial_m \bar{c} + \left[\, \langle\Sigma\rangle\,,\, \bar{c}\,\right], \tag{2.2.18}$$

then the combined supercharge $\widehat{\boldsymbol{Q}} \equiv \boldsymbol{Q} + \boldsymbol{Q}_{\mathrm{B}}$ acts on all the fields as

$$\widehat{\boldsymbol{Q}}^2 \;=\; i\pounds_v + \mathsf{Gauge}_{\langle\Sigma\rangle} - \frac{1}{2}\Big(\frac{1}{\bar{\ell}} + \frac{1}{\ell}\Big)\mathcal{R}_{U(1)}\,. \tag{2.2.19}$$

One may use $\widehat{\boldsymbol{Q}}$ as the localizing supercharge. We now move from $(A_m, \sigma, \lambda, \bar{\lambda}, D\,;\, c, \bar{c}, B)$ to cohomological variables with respect to $\widehat{\boldsymbol{Q}}$. They are given by 3 Grassmann-even plus 3 Grassmann-odd adjoint scalars

$$\begin{aligned} A_+ &\equiv \eta\gamma^m\xi A_m - i\eta\xi\sigma, \qquad & & c, \\ A_0 &\equiv \bar{\eta}\gamma^m\eta A_m + i\bar{\eta}\eta\sigma, \qquad & & \Lambda \equiv \eta\bar{\lambda} - \eta\bar{\lambda}, \\ A_- &\equiv \bar{\eta}\gamma^m\bar{\xi} A_m + i\bar{\eta}\bar{\xi}\sigma, \qquad & & \bar{c}, \end{aligned} \tag{2.2.20}$$

and their $\widehat{\boldsymbol{Q}} = (\boldsymbol{Q} + \boldsymbol{Q}_{\mathrm{B}})$-superpartners:

$$\begin{aligned} \boldsymbol{Q} A_+ &= i\xi\lambda, \qquad & \boldsymbol{Q}\, c &= i\delta\Sigma, \\ \boldsymbol{Q} A_0 &= i(\eta\bar{\lambda} + \eta\bar{\lambda}), \qquad & \boldsymbol{Q}\Lambda &= 2(D - H\sigma) + T, \\ \boldsymbol{Q} A_- &= i\bar{\eta}\bar{\lambda}, \qquad & \boldsymbol{Q}\,\bar{c} &= 0, \end{aligned} \tag{2.2.21}$$

$$\begin{aligned} \boldsymbol{Q}_{\mathrm{B}} A_+ &= \eta\gamma^m\xi\partial_m c - i[A_+\,,\, c] \qquad & \boldsymbol{Q}_{\mathrm{B}}\, c &= i c^2, \\ \boldsymbol{Q}_{\mathrm{B}} A_0 &= \bar{\eta}\gamma^m\eta\partial_m c - i[A_0\,,\, c] \qquad & \boldsymbol{Q}_{\mathrm{B}} \Lambda &= i[\, c\,,\, \Lambda], \\ \boldsymbol{Q}_{\mathrm{B}} A_- &= \bar{\eta}\gamma^m\bar{\xi}\partial_m c - i[A_-\,,\, c] \qquad & \boldsymbol{Q}_{\mathrm{B}}\, \bar{c} &= B, \end{aligned} \tag{2.2.22}$$

where

$$T = i(\eta\gamma^m\bar{\xi} - \bar{\eta}\gamma^m\xi)(\tilde{F}_m - K_m\sigma) + i(\eta\gamma^m\bar{\xi} + \bar{\eta}\gamma^m\xi)D_m\sigma. \tag{2.2.23}$$

It is straightforward to check that the change of variables is invertible and the Jacobian is trivial. Since we will add an appropriate localizing term to the action so that the Gaussian approximation is exact, it is enough to study $\widehat{\boldsymbol{Q}}$-transformation of the fields to the linear order in the fluctuations around the saddle points. And the problem becomes essentially the same as that of path-integral over matter fields coupled to a fixed vectormultiplet field. Under this approximation, the cohomological variables on the ellipsoid transform under $\widehat{\boldsymbol{Q}}$ as

$$\begin{aligned}
\widehat{\boldsymbol{Q}}\delta A_+ &\simeq i\xi\lambda + i\bar{\mathcal{J}}c, & \widehat{\boldsymbol{Q}}\bar{c} &= B,\\
\widehat{\boldsymbol{Q}}\delta A_- &\simeq i\bar{\xi}\bar{\lambda} - i\mathcal{J}c, & \widehat{\boldsymbol{Q}}c &\simeq -\delta\sigma + i\delta A_0,\\
\widehat{\boldsymbol{Q}}\delta A_0 &\simeq \frac{i}{2}(\bar{\xi}\lambda - \xi\bar{\lambda}) + \mathcal{L}_v c - i\big[v^m\langle A_m\rangle,\, c\big], & \widehat{\boldsymbol{Q}}\Lambda &\simeq 2\delta\Big(D - \frac{\sigma}{f}\Big) + \frac{4i}{f}\delta A_0,\\
& & &\quad + i\mathcal{J}\delta A_+ - i\bar{\mathcal{J}}\delta A_-
\end{aligned}$$

where $\simeq$ stands for the equality up to linear order in the fluctuation. This implies the relations among Hilbert spaces.

$$\mathcal{H}(A_+) \;\underset{\bar{\mathcal{J}}}{\overset{\mathcal{J}}{\rightleftarrows}}\; \mathcal{H}(A_0, c, \Lambda) \;\underset{\bar{\mathcal{J}}}{\overset{\mathcal{J}}{\rightleftarrows}}\; \mathcal{H}(A_-) \tag{2.2.24}$$

The one-loop determinant for a vectormultiplet is thus given by

$$\Delta^{\mathrm{v}}_{\text{1-loop}} = \left(\frac{\mathrm{Det}(\widehat{\boldsymbol{Q}}^2)_{\mathcal{H}(\bar{c})\oplus\mathcal{H}(c)\oplus\mathcal{H}(\Lambda)}}{\mathrm{Det}(\widehat{\boldsymbol{Q}}^2)_{\mathcal{H}(A_+)\oplus\mathcal{H}(A_-)\oplus\mathcal{H}(A_0)}}\right)^{\frac{1}{2}}. \tag{2.2.25}$$

Since $A_\pm$ have R-charge ± 2 and $A_0, \bar{c}, c, \Lambda$ have R-charge 0, this actually equals the one-loop determinant for an adjoint chiral multiplet with $r = 2$.

$$\begin{aligned}
\Delta^{\mathrm{v}}_{\text{1-loop}} &= \prod_{\alpha\in\Delta} s_b\Big(-\frac{iQ}{2} - \alpha\cdot\hat{\sigma}\Big)\\
&= \prod_{\alpha\in\Delta^+} 2\sinh(\pi b\alpha\cdot\hat{\sigma})\, 2\sinh(\pi b^{-1}\alpha\cdot\hat{\sigma}),
\end{aligned} \tag{2.2.26}$$

where Δ is the set of roots of G and Δ^+ is the set of positive roots.

2.3 The vortex operator on ellipsoid

Let us introduce the vortex loop operator introduced in Section 1.4 and evaluate its VEV by localization techniques.

If $\ell, \tilde{\ell}$ are incommensurable, there are only two circles on which closed loops of finite length along v can be wrapped. One is $S^1_{(\tau)}$ (the circle parametrized by τ) at $\theta = 0$, and the other is $S^1_{(\varphi)}$ at $\theta = \pi/2$. we will focus on a single loop operator wrapped on $S^1_{(\tau)}$ at $\theta = 0$.

The vortex loop wrapped on $S^1_{(\tau)}$ is defined by the gauge field behaving as

$$A \sim \beta d\varphi. \tag{2.3.1}$$

Solving $\boldsymbol{Q}\lambda = \boldsymbol{Q}\bar{\lambda} = 0$, one finds that the BPS condition is satisfied by setting the auxiliary field to be

$$D = iF_{12} - \frac{\sigma}{f}, \tag{2.3.2}$$

where $F_{12} = 2\pi\beta\delta^2(\theta = 0)$. This requires modifications of the saddle point configuration (2.1.2) on S^1_τ. Apparently it seems that the naive localization argument based on the vanishing of each term in the YM Lagrangian

$$0 = \mathrm{Tr}\left[\frac{1}{2}F_{mn}^2 + (D_m\sigma)^2 + (D - H\sigma)^2 + \cdots\right], \tag{2.3.3}$$

does not work. This is because the values of some fields are complex at the new saddle points. The supersymmetry preserved by (2.3.2) allows us to evaluate the VEV by applying the usual localization techniques without worry.

Now we move on the calculation of VEV of the vortex loop. The classical action, if defined as a naive volume integral, diverges due to the singular gauge field behavior. We thus should regularize the actions as discussed in Section 1.5. The regularized FI and CS actions on our saddle points are evaluated as

$$S_{\mathrm{FI}} + S_{\mathrm{FI,B}} = 2\pi i\zeta\ell\tilde{\ell}\left(\sigma + \frac{i\beta}{\tilde{\ell}}\right), \qquad S_{\mathrm{CS}} + S_{\mathrm{CS,B}} = -i\pi k\ell\tilde{\ell}\left(\sigma + \frac{i\beta}{\tilde{\ell}}\right)^2. \tag{2.3.4}$$

Note that the boundary at $\theta = \epsilon$ is oriented in such a way that $\int_{\theta=\epsilon} d\phi d\tau = -4\pi^2$. Similarly to the case without vortex loops, the regularized YM and matter kinetic actions vanish since they are $\boldsymbol{Q}$-exact.

From the above simple result for S_{cl}, one may guess that just replacing σ by $\sigma + \frac{i\beta}{\tilde{\ell}}$ in $\Delta_{\text{1-loop}}$ (2.2.15),(2.2.26) leads to the one-loop determinants in the presence of the vortex operator, but it is not so simple. In what follows we will explain it in detail.

One-loop determinants. As in the previous subsection, let us first consider the theory of a chiral multiplet of unit $U(1)$ charge, with the $U(1)$ vectormultiplet fields fixed at the saddle point, now in the presence of the vortex loop. $\Delta_{\text{1-loop}}$ can be easily computed by moving from (ϕ, ψ, F) to the cohomological variables (Φ, Ψ, Ψ', F'), and choosing a suitable localizing Lagrangian. The problem is thus reduced to finding the $\boldsymbol{Q}^2$-eigenmodes in the spaces ker$\mathcal{J}$ and coker$\mathcal{J}$.

$$\begin{aligned}
\mathcal{J} &= -ie^{-i(\varphi+\tau)}\left[-\frac{1}{f}\partial_\theta + \frac{i\cos\theta}{\tilde{\ell}\sin\theta}(\partial_\varphi - i\beta - irV_\varphi) - \frac{i\sin\theta}{\ell\cos\theta}(\partial_\tau - irV_\tau)\right], \\
\bar{\mathcal{J}} &= +ie^{+i(\varphi+\tau)}\left[-\frac{1}{f}\partial_\theta - \frac{i\cos\theta}{\tilde{\ell}\sin\theta}(\partial_\varphi - i\beta - i(r-2)V_\varphi) + \frac{i\sin\theta}{\ell\cos\theta}(\partial_\tau - i(r-2)V_\tau)\right].
\end{aligned} \tag{2.3.5}$$

Considering the behavior of the zeromode equation $\mathcal{J}\Phi = 0$, $\bar{\mathcal{J}}\Phi' = 0$ at $\theta = 0$ and $\pi/2$, we find

$$\begin{aligned}
\Phi = \hat{\Phi}(\theta)e^{im\varphi + in\tau} &\in \ker(\mathcal{J}) \implies \hat{\Phi}(\theta) \sim (\sin\theta)^{\beta - m}(\cos\theta)^{-n}, \\
\Phi' = \hat{\Phi}'(\theta)e^{im'\varphi + in'\tau} &\in \ker(\bar{\mathcal{J}}) \implies \hat{\Phi}'(\theta) \sim (\sin\theta)^{m' - \beta}(\cos\theta)^{n'},
\end{aligned} \tag{2.3.6}$$

where m, n, m', n' are integers.

In fact, in the presence of a vortex loop with non-integer β, not only the zeromodes (2.3.6) but all the eigenfunctions of $\bar{\mathcal{J}}\mathcal{J}$ or $\mathcal{J}\bar{\mathcal{J}}$, which are the natural basis wavefunctions of $\mathcal{H}$ or $\mathcal{H}'$, behave as fractional power of θ near $\theta = 0$ [29]. In this case, the simplest boundary condition requiring the wavefunctions of both $\mathcal{H}$ and $\mathcal{H}'$ to vanish at $\theta = 0$ is inconsistent for the following reason. In order for the $\boldsymbol{Q}$-transformation to be well-defined, the Hilbert spaces $\mathcal{H}$, $\mathcal{H}'$ need to satisfy

$$\mathcal{J}\mathcal{H} \subset \mathcal{H}', \qquad \bar{\mathcal{J}}\mathcal{H}' \subset \mathcal{H}\,. \tag{2.3.7}$$

Also, the operators $\mathcal{J}, \bar{\mathcal{J}}$ contain θ-derivatives which generically lower the power of θ by 1. Suppose a wavefunction $\Phi \in \mathcal{H}$ vanishes as θ^{γ} $(0 < \gamma < 1)$ near $\theta = 0$. Then $\mathcal{J}\Phi$, if nonzero, would have to be in $\mathcal{H}'$ and diverge as $\theta^{-(1-\gamma)}$ at $\theta = 0$. Similar argument holds with the role of $\mathcal{H}$ and $\mathcal{H}'$ exchanged.

As was proposed in [29] for a similar problem in two dimensions, there are two consistent boundary conditions for chiral multiplet fields at $\theta = 0$.

BC1. $\Phi \in \mathcal{H}$ is finite. $\Phi' \in \mathcal{H}'$ may diverge mildly but $\bar{\mathcal{J}}\Phi'$ is finite.

BC2. $\Phi' \in \mathcal{H}'$ is finite. $\Phi \in \mathcal{H}$ may diverge mildly but $\mathcal{J}\Phi$ is finite.

The mild divergence here means the behavior $\theta^{-\gamma}$ $(0 < \gamma < 1)$, which is not forbidden by the normalizability of wavefunctions. Note that *"is finite"* can be replaced by *"vanishes"* for non-integer β.

Let us compute $\Delta_{\text{1-loop}}$ for the chiral multiplet in the presence of a vortex loop. First, under the boundary condition **BC1**, the physical zeromodes of $\mathcal{J}, \bar{\mathcal{J}}$ are those in (2.3.6) with

$$\beta - m \geq 0\,, \quad -n \geq 0\,; \quad m' - \beta > -1\,, \quad n' \geq 0\,.$$

The first and the third inequalities are equivalent to $m \leq \lfloor \beta \rfloor$ and $m' \geq \lfloor \beta \rfloor$. These zeromodes all have definite $\boldsymbol{Q}^2$-eigenvalues which are now β-dependent. By multiplying all of them one obtains the one-loop determinant of a chiral multiplet on a vortex background:

$$\begin{aligned} \textbf{BC1} \quad \Longrightarrow \quad \Delta_{\text{1-loop}} &= \frac{\prod_{m' \geq \lfloor \beta \rfloor,\, n' \geq 0} \frac{m - \beta'}{\tilde{\ell}} + \frac{n'}{\ell} + i\sigma - \frac{r-2}{2}\left(\frac{1}{\tilde{\ell}} + \frac{1}{\ell}\right)}{\prod_{m \leq \lfloor \beta \rfloor,\, n \leq 0} \frac{m - \beta}{\tilde{\ell}} + \frac{n}{\ell} + i\sigma - \frac{r}{2}\left(\frac{1}{\tilde{\ell}} + \frac{1}{\ell}\right)} \\ &= s_b\Big(\frac{i(1-r)Q}{2} - \hat{\sigma} - ib\beta + ib\lfloor \beta \rfloor\Big)\,. \end{aligned} \tag{2.3.8}$$

The computation is similar for the boundary condition **BC2**. In this case, the integers m, m' in (2.2.9) are bounded as $\beta - m > -1$ and $m' - \beta \geq 0$, or equivalently $m \leq \lceil \beta \rceil$ and $m' \geq \lceil \beta \rceil$.

$$\begin{aligned} \textbf{BC2} \quad \Longrightarrow \quad \Delta_{\text{1-loop}} &= \frac{\prod_{m' \geq \lceil \beta \rceil,\, n' \geq 0} \frac{m - \beta'}{\tilde{\ell}} + \frac{n'}{\ell} + i\sigma - \frac{r-2}{2}\left(\frac{1}{\tilde{\ell}} + \frac{1}{\ell}\right)}{\prod_{m \leq \lceil \beta \rceil,\, n \leq 0} \frac{m - \beta}{\tilde{\ell}} + \frac{n}{\ell} + i\sigma - \frac{r}{2}\left(\frac{1}{\tilde{\ell}} + \frac{1}{\ell}\right)} \\ &= s_b\Big(\frac{i(1-r)Q}{2} - \hat{\sigma} - ib\beta + ib\lceil \beta \rceil\Big)\,. \end{aligned} \tag{2.3.9}$$

Note that $\Delta_{\text{1-loop}}$ is a periodic function of β for both boundary conditions. This is a consequence of large gauge invariance.

The above result can be easily generalized to the theory of chiral multiplet in a representation $\mathbf{R}$ of the gauge group G. The one-loop determinant is then given by a product over weights μ of $\mathbf{R}$.

$$\Delta^{\text{c}}_{\text{1-loop}}(\beta) = \prod_{\mu} s_b\Big(\frac{i(1-r)Q}{2} - \mu\cdot(\hat{\sigma} + ib\cdot\beta) + ib[\,\mu\cdot\beta]\Big). \tag{2.3.10}$$

Here $[\cdots]$ is the floor or ceiling functions depending on the choice of boundary condition.

Now let us move on to vectormultiplet. We already know that, as far as the one-loop determinant is concerned, a vectormultiplet is equivalent to an adjoint chiral multiplet with R-charge $r = 2$. The remaining question is which boundary condition should be chosen for that chiral multiplet, **BC1** or **BC2**.

In the presence of a vortex loop at $\theta = 0$, the Cartan part of $A_{+} = \eta\gamma^m\xi A_m$ and $A_{-} = \bar{\eta}\gamma^m\bar{\xi}A_m$:

$$\begin{aligned}
\eta\gamma^m\xi A_m &= e^{i(\varphi+\tau)}\left(\frac{1}{f}A_\theta + \frac{i\cos\theta}{\tilde{\ell}\sin\theta}A_\varphi - \frac{i\sin\theta}{\ell\cos\theta}A_\tau\right)\\
&\sim \frac{e^{i(\varphi+\tau)}}{\tilde{\ell}}\left(A_\theta + i\beta\theta^{-1}\right),\\
\bar{\eta}\gamma^m\bar{\xi}A_m, &= e^{-i(\varphi+\tau)}\left(\frac{1}{f}A_\theta - \frac{i\cos\theta}{\tilde{\ell}\sin\theta}A_\varphi + \frac{i\sin\theta}{\ell\cos\theta}A_\tau\right)\\
&\sim \frac{e^{-i(\varphi+\tau)}}{\tilde{\ell}}\left(A_\theta - i\beta\theta^{-1}\right),
\end{aligned} \tag{2.3.11}$$

diverge as θ^{-1} but $A_0 = \bar{\eta}\gamma^m\eta A_m$ is finite. It is therefore natural to allow mild divergence for $A_\pm$ but require A_0 to be finite at $\theta = 0$. Note that $A_\pm$ are the lowest components of the adjoint chiral multiplet with $r = 2$. In addition, the relation (2.2.24) says that c, Λ and $\bar{c}$ are finite. We thus conclude that the one-loop determinant of a vectormultiplet is equivalent to an adjoint chiral multiplet with R-charge $r = 2$ obeying **BC2**.

$$\Delta^{\text{v}}_{\text{1-loop}} = \prod_{\alpha\in\Delta} s_b\left(-\frac{iQ}{2} - \alpha\cdot(\hat{\sigma} + ib\beta) + ib\lfloor\alpha\cdot\beta\rfloor\right) \tag{2.3.12}$$

2.4 Partition function and vortex loop VEVs

Now we are ready to present exact formulae for the supersymmetric observables of our interest on an ellipsoid. First, the partition function can be expressed as [5]

$$Z_{S^3_b} \;=\; \frac{1}{|\mathcal{W}|}\int \mathrm{d}^r\hat{\sigma}\, e^{-S}\cdot\Delta^{\text{v}}_{\text{1-loop}}\cdot\Delta^{\text{c}}_{\text{1-loop}}, \tag{2.4.1}$$

where $r = \mathrm{rk}(G)$ and $\mathcal{W}$ is the Weyl group of G. S is the sum of the classical FI and CS actions evaluated at saddle points,

$$S_{\text{FI}} = 2\pi i\hat{\zeta}\hat{\sigma}, \qquad S_{\text{CS}} = -i\pi k\mathrm{Tr}\big(\hat{\sigma}^2\big), \tag{2.4.2}$$

where $\hat{\zeta} \equiv \sqrt{\ell\tilde{\ell}}\,\zeta$ is the dimensionless FI coupling. The one-loop determinants $\Delta^{\mathrm{c,v}}_{\text{1-loop}}$ in the absence of vortex loop take the form (2.3.10), (2.3.12).

The expectation value of a vortex loop can be expressed in a similar way,

$$\langle V_\beta\rangle = \frac{1}{|\mathcal{W}_K|}\int \mathrm{d}^r\hat{\sigma} e^{-S-S_{\mathrm{B}}}\cdot \Delta^{\mathrm{v}}_{\text{1-loop}}(\beta)\cdot \Delta^{\mathrm{c}\mathbf{1}}_{\text{1-loop}}(\beta)\cdot \Delta^{\mathrm{c}\mathbf{2}}_{\text{1-loop}}(\beta), \tag{2.4.3}$$

where $\mathcal{W}_K$ is the Weyl group of K (the centralizer of β) or equivalently the subgroup of $\mathcal{W}$ which leaves β invariant. We also separate the one-loop determinant of chiral multiplets according to the type of boundary conditions. We notice that the classical actions (2.3.4) remain the same as (2.4.2) if one redefines $\hat{\sigma}+ib\beta$ as $\hat{\sigma}$. Under the same redefinition of $\hat{\sigma}$, the one-loop determinants for vector and chiral multiplets become

$$\begin{aligned}
\Delta^{\mathrm{v}}_{\text{1-loop}}(\beta) &= \prod_{\alpha\in\Delta} s_b\Big(-\frac{iQ}{2} - \alpha\cdot\hat{\sigma} + ib\lceil\alpha\cdot\beta\rceil\Big),\\
\Delta^{\mathrm{c}\mathbf{1}}_{\text{1-loop}}(\beta) &= \prod_{\mu} s_b\Big(\frac{i(1-r)Q}{2} - \mu\cdot\hat{\sigma} + ib\lfloor\mu\cdot\beta\rfloor\Big),\\
\Delta^{\mathrm{c}\mathbf{2}}_{\text{1-loop}}(\beta) &= \prod_{\mu} s_b\Big(\frac{i(1-r)Q}{2} - \mu\cdot\hat{\sigma} + ib\lceil\mu\cdot\beta\rceil\Big).
\end{aligned} \tag{2.4.4}$$

Here we used the property of the double sine function (2.2.14). The one-loop determinants (2.4.4) essentially differ from those at $\beta = 0$ only by a product of sinh functions. Thus the expectation value of a vortex loop can be expressed as

$$\langle V_\beta\rangle \;=\; \frac{1}{|\mathcal{W}|}\int \mathrm{d}^r\hat{\sigma}\, e^{-S}\cdot \Delta^{\mathrm{v}}_{\text{1-loop}}\cdot \Delta^{\mathrm{c}}_{\text{1-loop}}\cdot V_\beta(\hat{\sigma}), \tag{2.4.5}$$

where S and $\Delta^{\mathrm{v,c}}_{\text{1-loop}}$ are the same as those for the partition function (2.4.1), and $V_\beta(\hat{\sigma})$ is the function which encodes the effects of insertion of a vortex loop.

$$\begin{aligned}
V_\beta(\hat{\sigma}) &= \frac{|\mathcal{W}|}{|\mathcal{W}_K|}\cdot V^{\mathrm{v}}_\beta(\hat{\sigma})\cdot V^{\mathrm{c}\mathbf{1}}_\beta(\hat{\sigma})\cdot V^{\mathrm{c}\mathbf{2}}_\beta(\hat{\sigma})\\
&= \frac{|\mathcal{W}|}{|\mathcal{W}_K|}\cdot \frac{\Delta^{\mathrm{v}}_{\text{1-loop}}(\beta)}{\Delta^{\mathrm{v}}_{\text{1-loop}}(0)}\cdot \frac{\Delta^{\mathrm{c}\mathbf{1}}_{\text{1-loop}}(\beta)}{\Delta^{\mathrm{c}\mathbf{1}}_{\text{1-loop}}(0)}\cdot \frac{\Delta^{\mathrm{c}\mathbf{2}}_{\text{1-loop}}(\beta)}{\Delta^{\mathrm{c}\mathbf{2}}_{\text{1-loop}}(0)}\,.
\end{aligned} \tag{2.4.6}$$

Note that, since we have redefined $\hat{\sigma}$, the contour of integration is now $\hat{\sigma} \in \mathfrak{h} + ib\beta$. In the following we will assume that it can be brought back to $\mathfrak{h}$ without problem. This is the case for pure YM-CS theories since $\Delta^{\mathrm{v}}_{\text{1-loop}}(\beta)$ has no poles. For theories with chiral multiplets this would lead to constraints on their R-charges r, representation $\mathbf{R}$ as well as β which we will not go into details.

Using Weyl group, generic β can be brought into a Weyl chamber so that $\alpha\cdot\beta > 0$ for all positive roots α. For non-generic β one has $\alpha\cdot\beta \geq 0$ for all positive roots α but $\alpha\cdot\beta = 0$ for some α, corresponding to the enhanced unbroken symmetry K. In what follows we make further simplifying assumption that β *is small*:

$$-1 < \alpha\cdot\beta < 1 \quad \text{for all roots} \quad \alpha, \qquad -1 < \mu\cdot\beta < 1 \quad \text{for all weights} \quad \mu. \tag{2.4.7}$$

Then $V_\beta(\hat\sigma)$ is the product of the following functions.

$$\begin{aligned}
V^{\mathrm{v}}_\beta(\hat\sigma) &= \prod_{\alpha\cdot\beta>0}\Big(2\sinh\pi b\alpha\cdot\hat\sigma\Big)^{-1}, \\
V^{\mathrm{c}\mathbf{1}}_\beta(\hat\sigma) &= \prod_{\mu\cdot\beta<0} 2\sinh\pi b\Big(\mu\cdot\hat\sigma+\frac{irQ}{2}\Big) \\
V^{\mathrm{c}\mathbf{2}}_\beta(\hat\sigma) &= \prod_{\mu\cdot\beta>0}\Big(2\sinh\pi b\Big(\mu\cdot\hat\sigma-\frac{i(2-r)Q}{2}\Big)\Big)^{-1}.
\end{aligned} \tag{2.4.8}$$

Here we neglected all the signs and powers of i's which can be absorbed into redefinition of the loop operator.

In the following sections we test the above formulae against some well-known facts. In fact, we will find that all these formulae need to be corrected.

Chapter 3

Pure Chern-Simons theories

The (bosonic) CS theory is a topological field theory which provides a physical description of a wide class of topological invariants associated to knots or links in 3-manifolds or the manifolds themselves. The theory was exactly solved in [41] by using non-perturbative methods and its relation to 2D conformal field theory with G symmetry.

$\mathcal{N}=2$ pure CS theories are essentially the same as the bosonic CS theories, because all the vectormultiplet fields except for the gauge field A_m are auxiliary fields. Some of the known formulae for observables in the bosonic CS theory can be reproduced using the results of the previous section. For example, the ellipsoid partition function of $\mathcal{N}=2$ CS theories is given by the following integral

$$\begin{aligned} Z &= \frac{1}{|\mathcal{W}|}\int \mathrm{d}^r\hat{\sigma}\, e^{-S_{\mathrm{CS}}}\cdot \Delta^{\mathrm{v}}_{\text{1-loop}} \\ &= \frac{1}{|\mathcal{W}|}\int \mathrm{d}^r\hat{\sigma}\, e^{i\pi k \mathrm{Tr}(\hat{\sigma}^2)} \prod_{\alpha\in\Delta^+} 2\sinh(\pi b\alpha\cdot\hat{\sigma})\cdot 2\sinh(\pi b^{-1}\alpha\cdot\hat{\sigma}). \end{aligned} \tag{3.0.1}$$

The result of [41] for the sphere partition function can be reproduced up to overall coefficients by setting $b=1$ and performing explicit $\hat{\sigma}$-integration with the help of Weyl's denominator formula

$$\prod_{\alpha\in\Delta^+} 2\sinh(\pi\alpha\cdot\hat{\sigma}) = \sum_{w\in\mathcal{W}} \epsilon(w) e^{2\pi w(\rho)\cdot\hat{\sigma}}, \tag{3.0.2}$$

where $\rho \equiv \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ is the Weyl vector and $\epsilon(w)=\pm 1$ is the parity of $w\in\mathcal{W}$. Likewise, the expectation value of an unknot can be reproduced as that of a BPS Wilson loop in $\mathcal{N}=2$ theory,

$$W_\Lambda(C) \equiv \mathrm{Tr}_\Lambda \mathrm{P}\exp i\oint_C (A_m v^m + i\sigma) a\mathrm{d}t\,, \tag{3.0.3}$$

where a is an arbitrary real constant and C is an integral curve of $\frac{\mathrm{d}}{\mathrm{d}t}x^m = av^m(x)$. As an example, take $C=S^1_{(\tau)}$ oriented in the increasing direction of τ (which is opposite to the direction of v^m). The Wilson loop expectation value is then given by an integral of the form (3.0.1) with an additional insertion of

$$W_\lambda(\widehat{\sigma}) = \mathrm{Tr}_\Lambda e^{2\pi\widehat{\sigma}} = \frac{\sum_{w\in\mathcal{W}}\epsilon(w)e^{2\pi w(\rho+\lambda)\cdot\widehat{\sigma}}}{\prod_{\alpha\in\Delta^+} 2\sinh\pi\alpha\cdot\widehat{\sigma}}. \tag{3.0.4}$$

Here λ is the highest weight of the representation Λ. Also, hereafter we will use a new dimensionless field $\widehat{\sigma}$

$$\widehat{\sigma} = b\hat{\sigma} = \ell\sigma, \tag{3.0.5}$$

which is more suitable than $\hat{\sigma}$ for the discussion of circular vortex loops of radius ℓ.

An important remark is in order. Many exact formulae for observables in bosonic CS theory depends on the CS coupling through the combination $k + h^\vee$, where $h^\vee$ is the dual Coxeter number of G. This can be understood as a perturbative correction at one-loop. But such shift of k does not occur in $\mathcal{N} = 2$ CS theories due to the presence of auxiliary fields [42]. Later we will encounter a similar difference between bosonic and $\mathcal{N} = 2$ theories concerning the shift of the label λ of Wilson loops [17].

3.1 Equivalence of Wilson and vortex loops

An interesting fact known in bosonic CS theories is that vortex loops are equivalent to Wilson loops in the representation with the highest weight $\lambda = k\beta/2$. We will first review how the equivalence works in bosonic CS theories, and then attempt to reproduce it in $\mathcal{N} = 2$ supersymmetric setting.

3.1.1 Quantization of (co)adjoint orbits

It is known that, for every irreducible representation Λ of a compact group G, there is a symplectic manifold (M, ω) which gives Λ as the Hilbert space of its geometric quantization. Using this, one can express a Wilson loop for arbitrary G and Λ by a suitable quantum mechanics on the loop interacting with the bulk gauge field. We summarize the basic idea here by going through one simple example. For more details of geometric quantization, see [43, 44].

Let us take $G = SU(2)$ and $\Lambda =$ spin-s representation. The symplectic manifold for this case is $M = S^2$ and the symplectic form $\omega = \hbar s \sin\theta \mathrm{d}\theta\mathrm{d}\varphi$, where θ, φ are the usual polar coordinates. We will keep the $\hbar$-dependence of various formulae for the next few paragraphs. The Hamiltonian functions (moment maps) and corresponding vector fields generating $SU(2)$ symmetry are given by

$$\begin{aligned}
P^1 &= -\hbar s\,\sin\theta\cos\varphi & X(P^1) &= -\sin\varphi\frac{\partial}{\partial\theta} - \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}, \\
P^2 &= -\hbar s\,\sin\theta\sin\varphi, & X(P^2) &= +\cos\varphi\frac{\partial}{\partial\theta} - \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}, \\
P^3 &= -\hbar s\,\cos\theta, & X(P^3) &= \frac{\partial}{\partial\varphi}.
\end{aligned} \tag{3.1.1}$$

They are related to each other by $\mathrm{d}P^a + \imath_{X(P^a)}\omega = 0$. The Poisson bracket on this M is defined by $\{\, f,\, g\} \equiv (\omega^{-1})^{mn}\,\partial_m f\,\partial_n g$. It satisfies

$$\{\varphi, \theta\} = \frac{1}{\hbar s\sin\theta}, \qquad \{P^a, P^b\} = \varepsilon^{abc}P^c.$$

In geometric quantization, Hilbert space is constructed in two steps. The first step, called *prequantization*, defines a map from functions $f, g, \cdots$ on M to operators $\hat{f}, \hat{g}, \cdots$ acting on certain Hilbert space $\mathcal{H}$ of wave functions by the formula

$$\hat{f} \equiv -i\hbar X(f) - \imath_{X(f)}\vartheta + f. \tag{3.1.2}$$

Here ϑ is a one-form satisfying $\mathrm{d}\vartheta = \omega$, which is necessary in order that $\{f_1, f_2\} = f_3$ lead to $[\hat{f}_1, \hat{f}_2] = i\hbar\hat{f}_3$[11]. But such a ϑ exists in general only locally. This makes the wave functions not ordinary functions on M but sections of a line bundle B, called *prequantum bundle*, with connection $\nabla = \mathrm{d} - i\hbar^{-1}\vartheta = \mathrm{d}x^m\nabla_m$. $\hat{f}$ is rewritten in term of the covariant derivative ∇_m as

$$\hat{f} = -i\hbar X(f)^m\nabla_m + f. \tag{3.1.3}$$

The symplectic form ω is then subject to the quantization condition

$$c_1(B) = \left[\frac{\omega}{2\pi\hbar}\right] \in H^2(M, \mathbb{Z}) .$$

In the present case it gives $\int_{S^2} \frac{\omega}{2\pi\hbar} = 2s \in \mathbb{Z}$.

The second step is to choose an integrable Lagrangian subbundle P of $TM^{\mathbb{C}}$ called *polarization* and require the *quantum* wave functions to be covariantly constant along $\bar{P}$. This is the generalization of the familiar fact that wave functions depend only on half of the phase space coordinates, and the complexification is to accommodate generalizations of coherent state quantization of harmonic oscillator. Various choices of P are possible for a given (M, ω), but for a Kähler manifold M a particularly convenient one is in which the quantum wave functions depend only on holomorphic coordinates. For the present example, $M = S^2$ can be covered by two coordinate patches $z = \tan\frac{\theta}{2}e^{i\varphi}$ and $w = \cot\frac{\theta}{2}e^{-i\varphi} = z^{-1}$. In the gauge

$$\vartheta_{[z]} = -2i\hbar s\frac{\bar{z}\,\mathrm{d}z}{1 + z\bar{z}}, \qquad \vartheta_{[w]} = -2i\hbar s\frac{\bar{w}\,\mathrm{d}w}{1 + w\bar{w}},$$

quantum wave functions Ψ are holomorphic functions in the respective coordinate patches. Moreover, $\Psi_{[z]}$ and $\Psi_{[w]}$ are related by $\Psi_{[w]} = z^{-2s}\Psi_{[z]}$, so they are both polynomials of degree $\leq 2s$. Quantum Hilbert space thus becomes $(2s + 1)$-dimensional as required for the spin-s representation.

The above simple problem can also be studied using path integral formalism [45]. The appropriate Lagrangian for the quantum mechanics of θ and φ is (hereafter we are back in $\hbar = 1$ units)

$$L = -s\cos\theta\dot{\varphi} + \gamma\dot{\varphi} , \tag{3.1.4}$$

where γ is a constant satisfying the quantization condition $s \pm \gamma \in \mathbb{Z}$.

Note that the first term in (3.1.4) gives the correct Poisson bracket of θ and φ in the same way that $\{q, p\} = 1$ follows from $L = p\dot{q}$. In other words, one has

$$\{\varphi, \pi_\varphi\} = 1, \qquad \pi_\varphi \equiv \frac{\partial L}{\partial\dot{\varphi}} , \tag{3.1.5}$$

[11] In Appendix B.1 we demonstrate this fact in detail.

and one can go to the quantum theory by replacing the above by the usual commutation relation $[\hat{\varphi}, \hat{\pi}_\varphi] = i$ (in $\hbar = 1$ units). In addition, the commutation relation of angular momenta $[\hat{J}^a, \hat{J}^b] = i\varepsilon^{abc}\hat{J}^c$ is reproduced by setting J^a:

$$J^1 = s\sin\theta\cos\varphi, \quad J^2 = s\sin\theta\sin\varphi, \quad J^3 = s\cos\theta, \tag{3.1.6}$$

which satisfy $\{J^a, J^b\} = \varepsilon^{abc}J^c$ as desired.

The second term in (3.1.4) and quantization condition for γ are necessary for $\exp(i\int \mathrm{d}tL)$ to be a continuous functional of the path $\{\theta(t), \varphi(t)\}$. It can be understood by thinking of continuous deformations of a path such that its winding number around the points $\theta = 0$ or π jumps.

For $\gamma = -s$ the above L and P^a can be expressed as

$$L = 2i\,\mathrm{Tr}(\lambda g^{-1}\dot{g}), \quad P^a = \mathrm{Tr}(\lambda g^{-1}\sigma^a g), \tag{3.1.7}$$

where σ^a are Pauli's matrices and λ, g are the following 2×2 matrices.

$$\lambda = \frac{s}{2}\sigma^3, \quad g = \exp\left(-\frac{i\varphi}{2}\sigma^3\right)\exp\left(-\frac{i\theta}{2}\sigma^2\right) = \begin{pmatrix} \sin\frac{\theta}{2} & e^{-i\varphi}\cos\frac{\theta}{2} \\ -e^{i\varphi}\cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}. \tag{3.1.8}$$

Using these quantities, one can express the Wilson loop as a path integral of a quantum mechanical system coupled to the 3D gauge field.

$$\begin{aligned} W_\lambda(C) &= \mathrm{Tr}_\Lambda \mathrm{P}\exp\left(i\oint_C \mathrm{d}x^m A^a_m T^a\right) \\ &= \int \mathcal{D}g \exp\int \mathrm{d}t\,\mathrm{Tr}\Big(-2\lambda g^{-1}(\dot{g} - i\dot{x}^m A_m g)\Big). \end{aligned} \tag{3.1.9}$$

The S^2 in the above discussion is the simplest example of *adjoint orbit*[12]. The adjoint orbit of a Lie algebra element $\lambda \in \mathfrak{g} = \mathrm{Lie}(G)$ is defined by

$$\mathrm{Ad}_G(\lambda) \equiv \{g\lambda g^{-1} \,|\, g \in G\}. \tag{3.1.10}$$

The irreducible representation of a Lie group with highest weight λ can be obtained from geometric quantization of the adjoint orbit $\mathrm{Ad}_G(\lambda)$, where the weight $\lambda \in \mathfrak{h}^*$ and the Lie algebra element $\lambda \in \mathfrak{h}$ are identified via

$$\lambda\cdot\sigma = 2\mathrm{Tr}(\lambda\sigma)\,. \qquad \big(\forall\sigma \in \mathfrak{h}\big) \tag{3.1.11}$$

The formula (3.1.9) works for arbitrary gauge groups and representations. General properties of adjoint orbits will be discussed in more detail later.

[12]Throughout this paper we work with the natural identification of adjoint and coadjoint orbits.

3.1.2 Boundary terms in CS theories revisited

In Chapter 1 we determined the boundary term for the CS action (1.5.7) from SUSY invariance. We are now in a position to argue this was not enough, and explain what needs to be added. Our argument is based on [16, 46] which carefully studied the canonical quantization of CS theories.

For simplicity, let us first consider the theory on $\mathbb{R}^3$ with a BPS vortex line satisfying (1.4.1), (1.4.4) lying along the x^3-axis. So M is an $\mathbb{R}^3$ with the tubular neighborhood of the vortex line removed. As is the previous chapter, we use t for the coordinate along the vortex line and the polar coordinate r, φ for the transverse two dimensions, so that ∂M is the cylinder at $r = \epsilon$ parameterized by φ, t. Our formula (1.5.7) for the boundary term for $\mathcal{N} = 2$ CS theory becomes in this case

$$S_{\mathrm{CS,B}} = -\frac{ik}{4\pi}\int_{\partial M} \mathrm{d}\varphi \mathrm{d}t\, \mathrm{Tr}\Big[A_\varphi(A_t - 2i\sigma)\Big]\,. \tag{3.1.12}$$

Let us examine if the variational problem is well-defined under this choice of boundary term.

Recall that the variation of the bosonic CS action gives

$$\begin{aligned} \delta S_{\mathrm{CS}} &= \delta\Big\{\frac{ik}{4\pi}\int_M \mathrm{Tr}\Big(A\mathrm{d}A - \frac{2i}{3}A^3\Big)\Big\} \\ &= \frac{ik}{2\pi}\int_M \mathrm{Tr}\big(\delta A\wedge F\big) + \frac{ik}{4\pi}\int_{\partial M}\mathrm{Tr}\big(\delta A\wedge A\big). \end{aligned} \tag{3.1.13}$$

The first term in the RHS vanishes due to the equation of motion $F = 0$. The second term can be rewritten as

$$\frac{ik}{4\pi}\int_{\partial M} \mathrm{d}\varphi \mathrm{d}t\, \mathrm{Tr}\Big(\delta A_\varphi A_t - A_\varphi \delta A_t\Big).$$

The variational problem becomes well-defined by requiring that one of the two gauge field components A_φ, A_t vanish on ∂M. Alternatively, one can specify nonzero boundary value for A_t by adding a boundary term

$$S_{\mathrm{CS,B}} = -\frac{ik}{4\pi}\int_{\partial M}\mathrm{d}\varphi\mathrm{d}t\,\mathrm{Tr}\big(A_\varphi A_t\big), \tag{3.1.14}$$

which is in fact a part of (3.1.12). Somewhat confusingly, the boundary term for specifying A_φ is different from this $S_{\mathrm{CS,B}}$ by minus sign. One can indeed check $\delta(S_{\mathrm{CS}} + S_{\mathrm{CS,B}})$ vanishes if $F = 0$ holds in the bulk and $\delta A_t = 0$ on the boundary.

$$\delta(S_{\mathrm{CS}} + S_{\mathrm{CS,B}}) = \frac{ik}{2\pi}\int_M \mathrm{Tr}(\delta A\wedge F) - \frac{ik}{2\pi}\int_{\partial M}\mathrm{d}\varphi\mathrm{d}\tau\ \mathrm{Tr}(A_\varphi \delta A_t) \tag{3.1.15}$$

Also, the addition of (3.1.14) has an effect of changing the bulk Lagrangian

$$\mathcal{L}_{\mathrm{CS}} = -\frac{ik}{4\pi}\mathrm{Tr}\Big(A_\varphi \dot{A}_t - A_t\dot{A}_\varphi + \cdots\Big) \quad\longrightarrow\quad \mathcal{L}'_{\mathrm{CS}} = -\frac{ik}{4\pi}\mathrm{Tr}\Big(2A_\varphi\dot{A}_t + \cdots\Big),$$

where the dots above A_t, A_φ stand for r-derivatives. Therefore, if the theory is radially quantized with the Lagrangian $\mathcal{L}'_{\mathrm{CS}}$, A_t plays the role of canonical coordinate and A_φ the momentum. The wave functions describing states on equal-r surfaces are functionals of A_t. This is in accord with the fact that one can set the value of A_t on the boundary at will.

Suppose that, instead of vortex singularity, a quantum mechanics with G symmetry is introduced along the x^3-axis. Let $S_{\rm QM}$ be the action describing the quantum mechanics interacting with the G-gauge field A_t in the bulk $\mathbb{R}^3$. Then one can define a 1D-3D coupled system by the path integral of $e^{-S_{\rm CS}-S_{\rm CS,B}-S_{\rm QM}}$ with respect to the quantum mechanical variables and the 3D gauge field. The boundary term which is appropriate for this construction is again (3.1.14).

Now that we have already chosen (3.1.14) as the boundary term, what can we do to impose the boundary condition on A_φ? The answer is simply to set

$$S_{\rm QM} = -ik\int {\rm d}t\,{\rm Tr}\big(\beta A_t\big). \tag{3.1.16}$$

Then the variation of the whole action

$$\delta\big(S_{\rm CS}+S_{\rm CS,B}+S_{\rm QM}\big) \;=\; \frac{ik}{2\pi}\int_M {\rm Tr}\big(\delta A\wedge F\big)-\frac{ik}{2\pi}\int_{\partial M}{\rm d}\varphi{\rm d}t\,{\rm Tr}\big(A_\varphi\delta A_t\big)-ik\int{\rm d}t\,{\rm Tr}\big(\beta\delta A_t\big)$$

gives $A_\varphi\big|_{\partial M}=\beta$ as an equation of motion. Furthermore, according to [16] one should average the boundary condition over the orbit of β, namely to modify the boundary condition as $A_\varphi\big|_{\partial M}=g\beta g^{-1}$ for a t-dependent element $g\in G$ and integrate over $g(t)$. This can be done by modifying $S_{\rm QM}$ as follows:

$$S_{\rm QM}[g] \;=\; k\int{\rm d}t\,{\rm Tr}\Big(\beta g^{-1}\Big(\frac{\rm d}{{\rm d}t}g-iA_tg\Big)\Big). \tag{3.1.17}$$

Here the kinetic term for $g(t)$ has been added to make $S_{\rm QM}$ gauge-invariant. We thus arrived at a description of vortex loops in terms of a quantum mechanics of $g(t)$ coupled to 3D gauge field. Moreover, the quantum mechanics is the same as the one for the Wilson loops (3.1.9) if their parameters λ,β are related as

$$\lambda=\frac{k\beta}{2}. \tag{3.1.18}$$

So, in bosonic CS theory with coupling k, a vortex loop with vorticity β is equivalent to a Wilson loop for the representation with the highest weight $\lambda=k\beta/2$. Note that this leads to a quantization of β in CS theories.

Let us come back to the $\mathcal{N}=2$ CS theories on an ellipsoid with a BPS vortex loop along $S^1_{(\tau)}$ at $\theta=0$. The supersymmetric boundary term is (1.5.7) instead of (3.1.12). The role of A_t,A_φ in the previous discussion is now played by

$$-v^mA_m=\frac{1}{\tilde\ell}A_\varphi+\frac{1}{\ell}A_\tau\,,\qquad \ell\tilde\ell\sin\theta\cos\theta\cdot w^mA_m=\ell\cos^2\theta A_\varphi-\tilde\ell\sin^2\theta A_\tau.$$

where w^m is defined in (1.5.6). To describe a vortex loop with vorticity β, one needs to introduce

$$S_{\rm QM}=k\int{\rm d}\tau{\rm Tr}\left[\beta\ell(iv^mA_m-\sigma)\right], \tag{3.1.19}$$

or the averaged version

$$S_{\rm QM}[g]=k\int{\rm d}\tau{\rm Tr}\left[\beta g^{-1}\frac{\rm d}{{\rm d}\tau}g+\beta g^{-1}\ell(iv^mA_m-\sigma)g\right]. \tag{3.1.20}$$

Note that we included σ in these formulae to make $S_{\rm QM}$ supersymmetric.

We believe that both of the above boundary terms lead to consistent descriptions of vortex loops. The boundary term $S_{\rm QM}$ (3.1.19) sets the boundary condition $A_\varphi = \beta$ and leads to the definition of a vortex loop by a singular behavior of the gauge field. On the other hand, the averaged version $S_{\rm QM}[g]$ describes a vortex loop in terms of a quantum mechanics coupled to the bulk gauge field. In the latter description of vortex loops, one usually does not assume singular behavior for the gauge field before integrating out the quantum mechanical degrees of freedom. These may sound somewhat empirical, but we would like to show in the following that the above two definitions indeed lead to the same result for the expectation value of a vortex loop.

3.1.3 Path integral over fields with singularity

Here we compute the expectation value of a BPS vortex loop on an ellipsoid using the boundary term without averaging, i.e. $S_{\rm QM}$ (3.1.19). Its value on the saddle point (2.2.2) and the boundary condition (2.3.1) is

$$S_{\rm QM} = -k\int \mathrm{d}\tau\, \mathrm{Tr}\Big[\beta\ell\Big(\sigma + \frac{i\beta}{\ell}\Big)\Big] = -2\pi k \mathrm{Tr}(\beta\widehat{\sigma}) = -\pi k\beta\cdot\widehat{\sigma}.$$

Note that we shifted σ as explained after (2.4.5) and then used (3.1.11). This corrects our previous formula for $V_\beta(\hat{\sigma})$ (2.4.6) and $V^{\rm v}_\beta(\hat{\sigma})$ (2.4.8) as follows:

$$V_\beta(\widehat{\sigma}) = \frac{|\mathcal{W}|}{|\mathcal{W}_K|}V^{\rm v}_\beta(\widehat{\sigma}), \qquad V^{\rm v}_\beta(\widehat{\sigma}) \;=\; \frac{e^{\pi k\beta\cdot\widehat{\sigma}}}{\prod_{\alpha\cdot\beta>0} 2\sinh\pi\alpha\cdot\widehat{\sigma}}\,. \tag{3.1.21}$$

Recall that β was gauge-rotated so that $\alpha\cdot\beta \geq 0$ for all the positive roots. Those which are orthogonal to β, if any, are the positive roots of the subgroup $K \subset G$ left unbroken by the vortex loop.

We would like to compare this with the function $W_\lambda(\widehat{\sigma})$ (3.0.4) for a Wilson loop in the representation Λ. We decompose the Weyl vector as $\rho = \rho_K + \tilde{\rho}$, where

$$\rho_K = \frac{1}{2}\sum_{\alpha\in\Delta_K^+}\alpha, \qquad \tilde{\rho} = \frac{1}{2}\sum_{\alpha\in\Pi^+}\alpha\,. \qquad \left(\begin{array}{l}\Delta_K^+ \equiv \{\alpha\in\Delta^+\,|\,\alpha\cdot\lambda = 0\}\\ \Pi^+ \equiv \{\alpha\in\Delta^+\,|\,\alpha\cdot\lambda > 0\}\end{array}\right) \tag{3.1.22}$$

Then

$$\begin{aligned} W_\lambda(\widehat{\sigma}) &= \frac{\sum_{w\in\mathcal{W}/\mathcal{W}_K}\sum_{w'\in\mathcal{W}_K}\epsilon(w)\epsilon(w')e^{2\pi w'(\rho+\lambda)\cdot w(\widehat{\sigma})}}{\prod_{\alpha\in\Delta^+}2\sinh\pi\alpha\cdot\widehat{\sigma}} \\ &= \sum_{w\in\mathcal{W}/\mathcal{W}_K}\frac{\sum_{w'\in\mathcal{W}_K}\epsilon(w')e^{2\pi(w'(\rho_K)+\tilde{\rho}+\lambda)\cdot w(\widehat{\sigma})}}{\prod_{\alpha\in\Delta^+}2\sinh\pi\alpha\cdot w(\widehat{\sigma})} \\ &= \sum_{w\in\mathcal{W}/\mathcal{W}_K} e^{2\pi(\tilde{\rho}+\lambda)\cdot w(\widehat{\sigma})}\,\frac{\prod_{\alpha\in\Delta_K^+}2\sinh\pi\alpha\cdot w(\widehat{\sigma})}{\prod_{\alpha\in\Delta^+}2\sinh\pi\alpha\cdot w(\widehat{\sigma})} \\ &= \sum_{w\in\mathcal{W}/\mathcal{W}_K}\frac{e^{2\pi(\tilde{\rho}+\lambda)\cdot w(\widehat{\sigma})}}{\prod_{\alpha\in\Pi^+}2\sinh\pi\alpha\cdot w(\widehat{\sigma})}\,, \end{aligned} \tag{3.1.23}$$

where $\mathcal{W}_K$ was defined at (2.4.3).

The expectation values of a Wilson loop in a representation with highest weight λ and a vortex loop with vorticity β are given respectively by integrals of $W_\lambda(\widehat{\sigma})$ and $V_\beta(\widehat{\sigma})$ over $\mathfrak{h}$ with a measure (2.4.5). Inside such an integral, the summation over the images of $\mathcal{W}/\mathcal{W}_K$ is the same as the multiplication by $|\mathcal{W}|/|\mathcal{W}_K|$. So the above result implies an equivalence between Wilson and vortex loops

$$V_\beta(\widehat{\sigma}) \simeq W_\lambda(\widehat{\sigma}) \qquad \text{for} \qquad \lambda + \tilde{\rho} = \frac{k\beta}{2}\,. \tag{3.1.24}$$

Note that there is a correction to the rule of correspondence compared to that for bosonic theory (3.1.18). This looks problematic because the trivial Wilson loop ($\lambda = 0$) does not correspond to the trivial vortex loop ($\beta = 0$).

3.2 1D-3D coupled system

Next we study the description of a vortex loop using the averaged version (3.1.20) of the boundary term. The quantization of (3.1.20) itself would give the representation with the highest weight $\lambda = k\beta/2$, because it is identical to the action (3.1.9) for the adjoint orbit quantization. We would like to do something slightly different here. As the bulk CS theory was promoted to a 3D $\mathcal{N} = 2$ theory, one can also promote the quantum mechanics on the vortex worldline to a 1D $\mathcal{N} = 2$ SUSY theory. The interaction between 1D and 3D fields can be chosen in such a way that the whole system is invariant under a SUSY that acts on both 1D and 3D fields at the same time. The path integral of the combined system can be performed exactly.

3.2.1 Adjoint orbits

We begin by summarizing basic properties of general adjoint orbits[13]. The adjoint orbit $M = \mathrm{Ad}_G(\lambda)$ for $\lambda \in \mathfrak{g}$ is defined by

$$\mathrm{Ad}_G(\lambda) \equiv \{g\lambda g^{-1} | g \in G\}. \tag{3.2.1}$$

M admits a *transitive action*, which means any two points on M are related by an element of G. For $g_1, g_2 \in G$, one point $g_1\lambda g_1^{-1} \in M$ maps to the other point $g_2\lambda g_2^{-1} \in M$ by the action of element $g_2 g_1^{-1} \in G$.

$$g_2^{-1}g_1 : \quad g_1\lambda g_1^{-1} \longmapsto g_2\lambda g_2^{-1}. \tag{3.2.2}$$

A manifold with a transitive action of Lie group is called a homogeneous manifold, which is identified with the coset space G/K. The group K is the stabilizer of a point. In the case $M = \mathrm{Ad}_G(\lambda)$, the stabilizer of a point λ is the centralizer of λ, namely the group K is the subgroup of G which consists of elements that commute with λ.

$$K = \{h \in G | h\lambda h^{-1} = \lambda\}. \tag{3.2.3}$$

[13] For more detailed reviews of the mathematical properties of adjoint orbits, see [47].

To describe mathematical properties of M, it is convenient to think of a map $g(x)$ ($x \in M$, $g \in G$) such as the $SU(2)$-valued function $g(\theta, \varphi)$ (3.1.8). The action of a Lie group element $g_0 \in G$ on M translates into a coordinate transformation $x^m \to x'^m$ according to the relation

$$g_0 \cdot g(x) = g(x') \cdot h(x, g_0) \qquad \big(h(x, g_0) \in K\big)\,, \tag{3.2.4}$$

since $h(x, g_0)$ commutes with λ and therefore

$$(g_0 g(x))\lambda(g_0 g(x))^{-1} = g(x')\lambda g(x')^{-1}. \tag{3.2.5}$$

As an infinitesimal version[14] of this, multiplication of Lie algebra generators $T^a \in \mathfrak{g}$ translates into the action of vector fields $X^a = X^{am}(x)\partial_m$,

$$X^a g(x) = -iT^a g(x) + ig(x)H^a(x)\,, \qquad \big(H^a(x) \in \mathfrak{k}\big) \tag{3.2.6}$$

where $\mathfrak{k} \subset \mathfrak{g}$ is the Lie algebra of K. The corresponding moment map function P^a is determined from $\mathrm{d}P^a + \imath_{X^a}\omega = 0$, where ω is the G-invariant symplectic form on M called the *Kirillov-Kostant-Souriau(KKS) 2-form.*

$$\omega = -2i\,\mathrm{Tr}\big[\lambda(g^{-1}\mathrm{d}g)^2\big]. \tag{3.2.7}$$

From $\mathrm{d}P^a = -\imath_{X^a}\omega$, we have

$$\begin{aligned}\imath_{X^a}\, 2i\mathrm{Tr}\big[\lambda \mathrm{d}g^{-1}\mathrm{d}g\big] &= 2i\,\mathrm{Tr}\big[\lambda(-iH^a g^{-1} + ig^{-1}T^a)\mathrm{d}g + \lambda \mathrm{d}g^{-1}(-iT^a g + igH^a)\big] \\ &= 2\mathrm{d}\,\left(\mathrm{Tr}\big[\lambda g^{-1}T^a g\big]\right),\end{aligned}$$

thus

$$P^a = 2\mathrm{Tr}\big[\lambda g^{-1}T^a g\big]. \tag{3.2.8}$$

One can easily check $\mathrm{d}\omega = 0$, and the G-invariance can be shown as follows.

$$\begin{aligned}g_0: \quad \omega \longmapsto 2i\,\mathrm{Tr}[\lambda\{(g_0 g h^{-1})^{-1}\mathrm{d}(g_0 g h^{-1})\}^2] &= 2i\,\mathrm{Tr}[\lambda(hg^{-1}\mathrm{d}g\cdot h^{-1} + h\mathrm{d}h^{-1})^2] \\ &= \omega\end{aligned}$$

In the above computation, after expanding the squared binomial inside the trace, one finds the $(h\mathrm{d}h^{-1})^2$ term is zero because of the fact that the 1-form $h\mathrm{d}h^{-1}$ takes values in $\mathfrak{k}$, the cross term is zero, and the remaining term is exactly ω. The G-invariance of ω can also be expressed as $\pounds_{X^a}\omega = 0$, which can be shown as follows.

$$\pounds_{X^a}\omega = \mathrm{d}\imath_{X^a}\omega = -\mathrm{d}^2 P^a = 0. \tag{3.2.9}$$

[14] Expanding (3.2.4) for an infinitesimal transformation with small parameter ϵ_a, one finds

$$\begin{aligned}g_0 g(x)h^{-1} &\backsimeq (1 + i\epsilon_a T^a)g(x)(1 - i\epsilon_a H^a(x)) \\ &= g(x) + i\epsilon_a T^a g(x) - i\epsilon_a g(x)H^a(x) + o(\epsilon^2).\end{aligned}$$

Next we turn to complex structures on M. Let $\mathfrak{n}$ be the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form. We are interested in the cases where G/K is reductive, that is when the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$ is such that $[\mathfrak{k}, \mathfrak{n}] \subset \mathfrak{n}$. Note that $\mathfrak{n}$ is identified with the tangent space at $\lambda \in M$. To define a complex structure J on M, one first needs a decomposition of $\mathfrak{n}^{\mathbb{C}}$ into two subspaces $\mathfrak{n}_{\pm}$ of definite eigenvalues of J. By transporting this decomposition of $T_\lambda M^{\mathbb{C}}$ to all other points on M by the action of G (3.2.4) one obtains an almost complex structure on M. It is integrable if the set of holomorphic (or antiholomorphic) vector fields on M is closed under Lie bracket, which simply amounts to $[\mathfrak{n}_{\pm}, \mathfrak{n}_{\pm}] \subset \mathfrak{n}_{\pm}$.

Recall that λ was chosen to be in a Cartan subalgebra $\mathfrak{h}$, and $\alpha \cdot \lambda \geq 0$ for all the positive roots $\alpha \in \Delta^+$. This leads to a decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$, where

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta_K^+} (\mathbb{C}E_\alpha + \mathbb{C}E_{-\alpha}), \qquad \mathfrak{n}_+ = \sum_{\alpha \in \Pi^+} \mathbb{C}E_\alpha, \qquad \mathfrak{n}_- = \sum_{\alpha \in \Pi^+} \mathbb{C}E_{-\alpha} \tag{3.2.10}$$

and Π^+ was defined in (3.1.22). The symplectic form (3.2.7) is of type $(1,1)$ under the complex structure thus defined, so M is a Kähler manifold. Note that there are in general multiple complex structures for a single coset space G/K. For example,

$$\lambda_1 = \mathrm{diag}(1,1,0,0,0,-1,-1), \qquad \lambda_2 = \mathrm{diag}(3,3,0,0,-2,-2,-2)$$

both break $G = SU(7)$ to $K = SU(3) \times SU(2)^2 \times U(1)^2$ but lead to different complex structures on G/K.

A useful fact is that G/K can be thought of as the *flag manifold* $G^{\mathbb{C}}/P$, where P is a *parabolic subgroup* of G corresponding to the Lie algebra $\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{n}_-$. (When K equals a maximal torus of G, P is called *Borel subgroup*.) This implies that any complex coordinate on N_+, the Lie group corresponding to $\mathfrak{n}_+$, can be used as a complex coordinate on M. Moreover, under such a choice of coordinate on M, the vector fields X^a (3.2.6) become holomorphic Killing vector fields which preserve the Kähler metric on M.

3.2.2 $\mathcal{N} = 2$ SUSY quantum mechanics on M

Let us now turn to the 1D $\mathcal{N} = 2$ supersymmetric quantum mechanics with the target space $M = \mathrm{Ad}_G(\lambda)$ and its quantization. As M is Kähler and we are gauging its isometry, we need chiral and vectormultiplets.

Take a complex coordinate z^I on M such that its metric and Kähler form are given in terms of the Kähler potential $K(z, \bar{z})$ as follows.

$$\mathrm{d}s^2 = g_{I\bar{J}}(z,\bar{z})\mathrm{d}z^I \mathrm{d}\bar{z}^{\bar{J}}, \quad \omega = i g_{I\bar{J}}(z,\bar{z})\mathrm{d}z^I \wedge d\bar{z}^{\bar{J}}; \qquad g_{I\bar{J}}(z,\bar{z}) = \frac{\partial^2 K(z,\bar{z})}{\partial z^I \partial \bar{z}^{\bar{J}}}. \tag{3.2.11}$$

The isometry of M is generated by holomorphic Killing vectors

$$X^a = X^{aI}(z)\frac{\partial}{\partial z^I} + \bar{X}^{a\bar{J}}(\bar{z})\frac{\partial}{\partial \bar{z}^{\bar{J}}}$$

satisfying $[X^a, X^b] = -f^{abc}X^c$. To each X^a there is a corresponding moment map P^a satisfying $\mathrm{d}P^a + \imath_{X^a}\omega = 0$, or in components

$$\partial_I P^a = i g_{I\bar{J}}\bar{X}^{a\bar{J}}, \qquad \bar{\partial}_{\bar{J}} P^a = -i g_{I\bar{J}} X^{aI}. \tag{3.2.12}$$

Using $g_{I\bar{J}} = \partial_I\bar{\partial}_{\bar{J}}K$ and the holomorphicity of Killing vector one can integrate these equalities to determine P^a up to constant shifts, which in turn can be fixed by requiring P^a to transform in the adjoint representation. For a suitable K, P^a can be written as

$$P^a = -iX^{aI}\partial_I K = i\bar{X}^{a\bar{J}}\bar{\partial}_{\bar{J}}K\,. \tag{3.2.13}$$

A (1D) vectormultiplet consists of a gauge field A_t, bosons σ, D and fermions $\lambda, \bar{\lambda}$ transforming as

$$\begin{aligned}
\boldsymbol{Q}A_t &= \frac{i}{2}(\epsilon\bar{\lambda} + \bar{\epsilon}\lambda), & \boldsymbol{Q}\lambda &= \epsilon(-iD_t\sigma - D),\\
\boldsymbol{Q}\sigma &= \frac{1}{2}(\epsilon\bar{\lambda} + \bar{\epsilon}\lambda), & \boldsymbol{Q}\bar{\lambda} &= \bar{\epsilon}(-iD_t\sigma + D),\\
\boldsymbol{Q}D &= -\frac{i}{2}D_t(\epsilon\bar{\lambda} - \bar{\epsilon}\lambda) + \frac{i}{2}[\sigma, \epsilon\bar{\lambda} - \bar{\epsilon}\lambda], & &
\end{aligned} \tag{3.2.14}$$

where $\epsilon, \bar{\epsilon}$ are Grassmann-even constant SUSY parameters. All the fields are Lie algebra valued, so one can express them using the set of generators T^a as follows.

$$A_t = A_t^a T^a, \quad \sigma = \sigma^a T^a, \quad \text{etc.} \qquad \Big([T^a, T^b] = if^{abc}T^c\Big)$$

The complex coordinates z^I on M are promoted to chiral multiplets. Each chiral multiplet consists of a boson z^I and its superpartner χ^I. They transform as

$$\begin{aligned}
\boldsymbol{Q}z^I &= \epsilon\chi^I, & \boldsymbol{Q}\chi^I &= -i\bar{\epsilon}\big(D_t z^I - i\sigma^a X^{aI}\big), & D_t z^I &\equiv \dot{z}^I + A_t^a X^{aI},\\
\boldsymbol{Q}\bar{z}^{\bar{J}} &= \bar{\epsilon}\bar{\chi}^{\bar{J}}, & \boldsymbol{Q}\bar{\chi}^{\bar{J}} &= -i\epsilon\big(D_t\bar{z}^{\bar{J}} - i\sigma^a\bar{X}^{a\bar{J}}\big), & D_t\bar{z}^{\bar{J}} &\equiv \dot{\bar{z}}^{\bar{J}} + A_t^a\bar{X}^{a\bar{J}}.
\end{aligned} \tag{3.2.15}$$

The SUSY-invariant kinetic Lagrangian for the chiral multiplets is given by

$$\begin{aligned}
L_{\text{kin}} \;=\;& g_{I\bar{J}}D_t\bar{z}^{\bar{J}}D_t z^I + g_{I\bar{J}}\bar{X}^{a\bar{J}}X^{bI}\sigma^a\sigma^b + iD^aP^a - g_{I\bar{J}}\bar{X}^{a\bar{J}}\lambda^a\chi^I + g_{I\bar{J}}\bar{\chi}^{\bar{J}}\bar{\lambda}^a X^{aI}\\
&- ig_{I\bar{J}}\bar{\chi}^{\bar{J}}D_t\chi^I + g_{I\bar{J}}\bar{\chi}^{\bar{J}}\partial_K X^{aI}\sigma^a\chi^K + g_{I\bar{J},K}\bar{\chi}^{\bar{J}}X^{aK}\sigma^a\chi^I,\\
&D_t\chi^I \equiv \dot{\chi}^I + A_t^a\partial_K X^{aI}\chi^K + \Gamma^I_{KL}D_t z^K\chi^L.
\end{aligned} \tag{3.2.16}$$

Another invariant can be constructed using the one-form $\vartheta = \vartheta_I \mathrm{d}z^I + \vartheta_{\bar{J}}\mathrm{d}\bar{z}^{\bar{J}}$ satisfying $\mathrm{d}\vartheta = \omega$.

$$\begin{aligned}
L_{\text{top}} &= ig_{I\bar{J}}\chi^I\bar{\chi}^{\bar{J}} - i\vartheta_I\big(D_t z^I - i\sigma^a X^{aI}\big) - i\vartheta_{\bar{J}}\big(D_t\bar{z}^{\bar{J}} - i\sigma^a\bar{X}^{a\bar{J}}\big)\\
&= ig_{I\bar{J}}\chi^I\bar{\chi}^{\bar{J}} - i(\vartheta_I\dot{z}^I + \vartheta_{\bar{J}}\dot{\bar{z}}^{\bar{J}}) - (\sigma^a + iA_t^a)P^a.
\end{aligned} \tag{3.2.17}$$

What we actually need to do is to gauge the isometry of the adjoint orbit M by the 3D gauge field and not by an independent 1D vector field. To do this in a supersymmetric manner, we recall the transformation rule of cohomological variables constructed from the 3D vectormultiplet fields.

$$\begin{aligned} \boldsymbol{Q}(u^m A_m) &= \frac{i}{2}(\eta\bar{\lambda} + \bar{\eta}\lambda), & \boldsymbol{Q}(\bar{\eta}\lambda) &= -iu^m D_m\sigma - \tilde{D}, \\ \boldsymbol{Q}\sigma &= \frac{1}{2}(\eta\bar{\lambda} + \bar{\eta}\lambda), & \boldsymbol{Q}(\eta\bar{\lambda}) &= -iu^m D_m\sigma + \tilde{D}, \\ \boldsymbol{Q}\tilde{D} &= -\frac{i}{2}u^m D_m(\eta\bar{\lambda} - \bar{\eta}\lambda) + \frac{i}{2}[\sigma, \eta\bar{\lambda} - \bar{\eta}\lambda]. \end{aligned} \tag{3.2.18}$$

Here $u^m \equiv \bar{\eta}\gamma^m\eta$ is equal to $-v^m$ on the ellipsoid and $\tilde{D} \equiv D - \frac{1}{f}\sigma - u^m\tilde{F}_m$. By comparing this with (3.2.14) one finds that the 3D fields

$$u^m A_m\ ,\ \sigma\ ,\ \bar{\eta}\lambda\ ,\ \eta\bar{\lambda}\ ,\ \tilde{D}$$

transform under the 3D SUSY in the same way that the 1D vectormultiplet transforms under 1D SUSY with $\epsilon = \bar{\epsilon} = 1$. The 1D-3D coupling is thus obtained by identifying t with $\ell\tau$, replacing the vectormultiplet fields in (3.2.16), (3.2.17) by the above 3D fields and regarding $\boldsymbol{Q}^{(3\mathrm{D})} + \boldsymbol{Q}^{(1\mathrm{D})}_{(\epsilon=\bar{\epsilon}=1)}$ as the SUSY of the total system. Recalling (3.2.8) and (3.2.7) one finds that the bosonic part of L_{top} (3.2.17) agrees precisely with the action $S_{\mathrm{QM}}[g]$ (3.1.20) for the quantum mechanics on vortex loops, and the fermions appear in L_{top} as auxiliary fields.

The Lagrangians L_{top} and L_{kin} play a role similar to that of S_{CS} and S_{YM} for the 3D gauge field. First, the fermions $\chi^I, \bar{\chi}^{\bar{J}}$ are auxiliary variables in the theory without L_{kin}. Second, $L_{\mathrm{top}} = \boldsymbol{Q}\Psi_{\mathrm{top}}$ but Ψ_{top} depends on the components of ϑ

$$\Psi_{\mathrm{top}} = \vartheta_I\chi^I + \vartheta_{\bar{J}}\bar{\chi}^{\bar{J}},$$

which are defined only up to (Kähler) gauge transformations. As a consequence, L_{top} takes different nonzero values on different saddle points, whereas L_{kin} vanishes at every saddle point.

Witten index. Let us compute the Witten index, i.e. the S^1 partition function of the quantum mechanics on a vortex loop. It is a SUSY quantum mechanics with the target space $\mathcal{M} = \mathrm{Ad}_G(\lambda)$ coupled to 3D vectormultiplet field. The 3D fields are fixed at a saddle point (2.2.2). So we only need to study the 1D theory defined by (3.2.15), (3.2.16) and (3.2.17) with all the vectormultiplet fields turned off except for constant σ, which we may assume to be in $\mathfrak{h}$.

According to (3.2.15), the saddle point condition for our quantum mechanics is

$$\dot{z}^I - i\sigma^a X^{aI} = 0, \qquad \dot{\bar{z}}^{\bar{J}} - i\sigma^a \bar{X}^{a\bar{J}} = 0.$$

In terms of the original coordinate g on M, these become

$$\frac{\mathrm{d}}{\mathrm{d}t}(g\lambda g^{-1}) - [\sigma, g\lambda g^{-1}] = 0.$$

If σ and the periodicity of t take generic values, this can only be solved by requiring the two terms on the LHS vanish independently. So, $g\lambda g^{-1}$ is a constant element of $\mathfrak{h}$ at saddle points. Since λ is also an element of $\mathfrak{h}$, $g\lambda g^{-1}$ has to be an image of λ under Weyl group.

Let us study the saddle point $g = \mathrm{id}$ ($g\lambda g^{-1} = \lambda$) in detail. The neighborhood of this point can be covered by a local complex coordinate system z^α such that

$$g = \exp i \sum_{\alpha\in\Pi^+} (z^\alpha E_\alpha + \bar{z}^\alpha E_{-\alpha}). \tag{3.2.19}$$

Then the Kähler form and metric around this point are approximately given by

$$\omega \simeq i \sum_{\alpha\in\Pi^+} g_{\alpha\bar{\alpha}} \mathrm{d}z^\alpha \wedge \mathrm{d}\bar{z}^\alpha, \qquad g_{\alpha\bar{\alpha}} \equiv 2\lambda\cdot\alpha \,\mathrm{Tr}\big(E_\alpha E_{-\alpha}\big). \tag{3.2.20}$$

Note the positive definiteness of the metric. The moment map and the Killing vector corresponding to $\sigma = \sigma_i H_i \in \mathfrak{h}$ read

$$\sigma_i P_i \simeq \lambda\cdot\sigma - \sum_{\alpha\in\Pi^+} (\alpha\cdot\sigma) z^\alpha \bar{z}^\alpha g_{\alpha\bar{\alpha}}, \qquad \sigma_i X_i = -i \sum_{\alpha\in\Pi^+} \alpha\cdot\sigma\Big(z^\alpha \frac{\partial}{\partial z^\alpha} - \bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha}\Big), \tag{3.2.21}$$

where we used $[H_i, E_\alpha] = \alpha_i E_\alpha$. Note that the expression for the Killing vector is exact. The value of the action (the integral of L_{top}) on this saddle point is

$$e^{-S_{\mathrm{QM}}} = e^{2\pi\ell\lambda\cdot\sigma} = e^{2\pi\lambda\cdot\widehat{\sigma}}. \tag{3.2.22}$$

The one-loop determinant $\Delta_{\text{1-loop}}$ at this saddle point can be computed using the SUSY-exact localizing Lagrangian L_{kin}, which takes the approximate form

$$L_{\mathrm{kin}} \simeq \sum_{\alpha\in\Pi^+} g_{\alpha\bar{\alpha}} \Big\{ \dot{\bar{z}}^\alpha \dot{z}^\alpha + (\alpha\cdot\sigma)^2 \bar{z}^\alpha z^\alpha - i\bar{\chi}^\alpha \dot{\chi}^\alpha - i(\alpha\cdot\sigma)\bar{\chi}^\alpha \chi^\alpha \Big\}. \tag{3.2.23}$$

The Gaussian integration over z^α and χ^α can be easily performed using $\det\big(\frac{\mathrm{d}}{\mathrm{d}t} + \omega\big) = 2\sinh \pi\ell\omega$ (if $t \sim t + 2\pi\ell$). The contribution of this saddle point finally becomes

$$e^{-S_{\mathrm{QM}}} \cdot \Delta_{\text{1-loop}} \Big|_{g\lambda g^{-1}=\lambda} \;=\; \frac{e^{2\pi\lambda\cdot\widehat{\sigma}}}{\prod_{\alpha\in\Pi^+} 2\sinh \pi\alpha\cdot\widehat{\sigma}}. \tag{3.2.24}$$

Other saddle points are all characterized by the equation $g\lambda g^{-1} = w(\lambda)$ for some element w of the Weyl group. Their contribution can be computed by repeating the above steps with the replacement $\lambda \to w(\lambda)$. But once this replacement is made, the set of positive roots also needs to be redefined so that $\alpha\cdot w(\lambda) \geq 0$ for all $\alpha \in \Delta^+_{(\mathrm{new})}$. So the contribution from other saddle points are obtained from (3.2.24) by replacing $\lambda \to w(\lambda)$ and $\alpha \to w(\alpha)$, or more simply by the replacement $\widehat{\sigma} \to w^{-1}(\widehat{\sigma})$. The full partition function is thus obtained by summing over different saddle points labeled by $w \in \mathcal{W}/\mathcal{W}_K$. The index finally becomes

$$\begin{aligned} I_\lambda(\widehat{\sigma}) \;&=\; \int \mathcal{D}[z,\chi] \exp\left(-S_{\mathrm{QM}}\right) \\ &=\; \sum_{w\in\mathcal{W}/\mathcal{W}_K} \frac{e^{2\pi\lambda\cdot w(\widehat{\sigma})}}{\prod_{\alpha\in\Pi^+} 2\sinh \pi\alpha\cdot w(\widehat{\sigma})}. \end{aligned} \tag{3.2.25}$$

Comparison of the results. The functions $V_\beta(\widehat{\sigma})$ (3.1.21) and $I_\lambda(\widehat{\sigma})$ are to be integrated over $\mathfrak{h}$ with a $\mathcal{W}$-invariant measure (2.4.5) to give the expectation value of a vortex loop defined in two different ways. Taking account of the fact that the sum over the Weyl images is redundant inside the integral, one finds

$$V_\beta(\widehat{\sigma}) \simeq I_\lambda(\widehat{\sigma}) \qquad \text{for} \qquad \lambda = \frac{k\beta}{2}. \tag{3.2.26}$$

This gives a precise correspondence between the two definitions of a BPS vortex loop, namely the boundary condition $A_\varphi = \beta$ versus an $\mathcal{N} = 2$ SUSY quantum mechanics with the target $M = \mathrm{Ad}_G(\lambda)$.

On the other hand, the insertion of a BPS Wilson loop in the representation with the highest weight λ is described by $\mathrm{Tr}_\lambda e^{2\pi\widehat{\sigma}} = W_\lambda(\widehat{\sigma})$ (3.1.23). This function can be reproduced from a *non-supersymmetric* quantum mechanics with the target $M = \mathrm{Ad}_G(\lambda)$ and the action (3.1.9). Our computation shows that the partition functions of the bosonic and supersymmetric quantum mechanics with the same target $M = \mathrm{Ad}_G(\lambda)$ are slightly different:

$$I_\lambda(\widehat{\sigma}) = W_{\lambda-\tilde{\rho}}(\widehat{\sigma})\,. \tag{3.2.27}$$

Similar shift of parameter was noticed and studied in some earlier works [17, 48]. This result may look strange since the bosonic model was supersymmetrized by adding fermions as auxiliary fields. However, when computing I_λ we perturbed the theory further by L_{kin}, and as a consequence the fermions became dynamical. In fact, the problem is similar to the evaluation of perturbative correction to the CS coupling of SUSY YM-CS theory [42]. For the simplest case $G = SU(2)$ it was shown by an explicit one loop analysis that the added fermions give rise to a shift of the spin s by $-1/2$ [17].

3.3 Resolution of the unwanted parameter shift

As we have seen, there is a subtle difference between the bosonic and $\mathcal{N} = 2$ theories which appears as the shift $\lambda \to \lambda - \tilde{\rho}$ in the formulae for observables. Here we would like to argue that one can (and should) nevertheless relate the Wilson and vortex loops in $\mathcal{N} = 2$ theory by the same formula $\lambda = k\beta/2$ as in bosonic theory. For this purpose, we need to explain the effect of the added fermions in more detail.

It is worth noting that the partition function I_λ of the $\mathcal{N} = 2$ SUSY quantum mechanics agrees precisely with that of geometric quantization with the so-called *metaplectic correction* taken into account. The importance of metaplectic correction is often skipped over, but when applied to the system of harmonic oscillator, it gives the correct account of its zero-point energy from the requirement of internal consistency alone. The origin of the metaplectic correction can be understood by studying how the quantum Hilbert spaces corresponding to different polarizations are related to each other, and in particular how the group of canonical transformations (the symplectic group) is represented. See for example [43] for more detail. The upshot is that, if the quantum Hilbert spaces are constructed from the space of sections of the prequantum bundle B, the symplectic group will be represented only projectively. But it can be improved by replacing B by $B \otimes K^{1/2}$, where K is the canonical bundle of the target space M. Note that K does not always have a well-defined square root, and $K^{1/2}$ may not be unique even if it exists.

Let us calculate the metaplectic correction for the case $M = \mathrm{Ad}_G(\lambda)$. Since the correction should preserve the property of M as a homogeneous manifold with G-symmetry, it should at most modify the parameter λ. Take a function $\sigma_i P_i$ and the vector field $\sigma_i X_i$ in (3.2.21), and consider the action of the corresponding operator $\sigma_i \hat{P}_i$ on quantum wave functions in the holomorphic polarization. Before the metaplectic correction, $\sigma_i \hat{P}_i$ is the following differential operator near $z^\alpha = 0$.

$$\begin{aligned} \sigma_i \hat{P}_i &= -i \sum_{\alpha \in \Pi^+} \sigma_i X_i^\alpha \Big(\frac{\partial}{\partial z^\alpha} - i\vartheta_\alpha \Big) + \sigma_i P_i \\ &= \lambda \cdot \sigma - \sum_{\alpha \in \Pi^+} (\alpha \cdot \sigma) z^\alpha \frac{\partial}{\partial z^\alpha}. \end{aligned} \tag{3.3.1}$$

After the metaplectic correction, wave functions transform differently under infinitesimal coordinate transformations. So the definition of the operator is also modified accordingly.

$$\begin{aligned} \sigma_i \hat{P}_i &= -i \sum_{\alpha \in \Pi^+} \Big[\sigma_i X_i^\alpha \Big(\frac{\partial}{\partial z^\alpha} - i\vartheta_\alpha \Big) + \frac{1}{2} \frac{\partial(\sigma_i X_i^\alpha)}{\partial z^\alpha} \Big] + \sigma_i P_i \\ &= \lambda \cdot \sigma - \frac{1}{2} \sum_{\alpha \in \Pi^+} (\alpha \cdot \sigma) - \sum_{\alpha \in \Pi^+} (\alpha \cdot \sigma) z^\alpha \frac{\partial}{\partial z^\alpha}. \end{aligned} \tag{3.3.2}$$

This shows that the shift $\lambda \to \lambda - \tilde{\rho}$ can indeed be explained by metaplectic correction.

Another important effect of the fermions in $\mathcal{N} = 2$ SUSY quantum mechanics is the global anomaly [18]. The fact that the highest weight λ receives quantum correction implies that the G-symmetry of the quantum mechanics may be anomalous, because $\lambda - \tilde{\rho}$ is not always a weight of G. The anomaly arises from quantization of the fermions. Consider a theory with fermions $\chi, \bar{\chi}$ valued in linear spaces $V_{\mathrm{F}}, V_{\mathrm{F}}^*$ and a Lagrangian of the form

$$L = i\bar{\chi} D_t \chi + \cdots . \tag{3.3.3}$$

Quantization of the fermions leads to the Hilbert space of fermionic states

$$\mathcal{H}_{\mathrm{F}} = \det{}^{-\frac{1}{2}} V_{\mathrm{F}} \otimes \wedge V_{\mathrm{F}}. \tag{3.3.4}$$

If V_{F} represents a symmetry, then the symmetry has an anomaly unless $\det^{\frac{1}{2}} V_{\mathrm{F}}$ gives a well-defined one-dimensional representation. For $\mathcal{N} = 2$ SUSY non-linear sigma model (NLSM) with the target space M discussed in Section 3.2.2, the fermions χ take values on the pull back of the holomorphic tangent bundle T_M by the boson z. The Hilbert space of this model is thus identified with the space of sections of the bundle

$$K^{1/2} \otimes \wedge T_M \otimes B. \tag{3.3.5}$$

The model has an anomaly unless this is a well-defined vector bundle. Note the similarity of (3.3.5) with the metaplectic correction. As an example, for the case $M = S^2$ with $\omega =$

$s\sin\theta \mathrm{d}\theta\mathrm{d}\varphi$ one can show by canonical quantization that the Hilbert spaces of the bosonic and $\mathcal{N}=2$ supersymmetric NLSMs are spanned by monopole harmonics [17]. They can therefore be decomposed into irreducible representations of $SU(2)$:

$$\begin{aligned}\mathcal{H}_{\mathcal{N}=0} &= \bigoplus_{n\in\mathbb{Z}_{\geq 0}} (\text{spin } s+n),\\ \mathcal{H}_{\mathcal{N}=2} &= \Big[\bigoplus_{n\in\mathbb{Z}_{\geq 0}} (\text{spin } s-\tfrac{1}{2}+n)\Big]_{\text{boson}} \oplus \Big[\bigoplus_{n\in\mathbb{Z}_{\geq 0}} (\text{spin } s+\tfrac{1}{2}+n)\Big]_{\text{fermion}}.\end{aligned} \tag{3.3.6}$$

Note that these Hilbert spaces are for NLSMs which have a mixture of the first and second order kinetic terms for bosons. As the second order kinetic term is turned off, only the representation with the lowest spin remains and others are all lifted up to extremely high energy. This is another way to see the shift $s \to s-1/2$.

The global anomaly in $\mathcal{N}=2$ SUSY quantum mechanics can be canceled by turning on a suitable Wilson line [18]. This is because the introduction of a Wilson line with charge q,

$$\exp\left(-\int \mathrm{d}t L_{\mathrm{WL}}\right) = \exp\left(iq\int \mathrm{d}t A_t\right),$$

has an effect to shift the charge of all the states uniformly by q. In fact, L_{top} (3.2.17) can be regarded as a Wilson line in which the pull back of ϑ plays the role of A_t. This can be used to cancel the unwanted shift of λ while maintaining the relation $\lambda = k\beta/2$. We define the BPS vortex loop with vorticity β by a 1D $\mathcal{N}=2$ SUSY quantum mechanics with the target $M = \mathrm{Ad}_G(\lambda)$, $\lambda = k\beta/2$ and the Wilson line which precisely cancels the shift $\lambda \to \lambda - \tilde{\rho}$. As we will see in the next chapter, this definition turns out to be more convenient when describing the quantum mechanics on vortex loops in terms of gauged linear sigma models.

An example: $\mathbb{CP}^{N-1}$**.** We close this chapter with one concrete example. Take $G = SU(N)$ and

$$\lambda = m\left(\frac{N-1}{N}, -\frac{1}{N}, \cdots, -\frac{1}{N}\right) \in \mathfrak{h}^*, \qquad k\beta = m\cdot\mathrm{diag}\Big(\frac{N-1}{N}, -\frac{1}{N}, \cdots, -\frac{1}{N}\Big) \in \mathfrak{h}. \tag{3.3.7}$$

The corresponding adjoint orbit is $\mathbb{CP}^{N-1}$ with the prequantum bundle $B = \mathcal{O}(m)$. The quantum mechanical partition function is supposed to reproduce the character for the M-th symmetric tensor representation of $SU(N)$.

We start from the Euclidean action (3.1.20) for the vortex loop along $S^1_{(\tau)}$ at $\theta = 0$:

$$S \;=\; k\int \mathrm{d}\tau \mathrm{Tr}\Big[\beta g^{-1}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau} - iA_\tau - \widehat{\sigma}\Big)g\Big]. \tag{3.3.8}$$

We assume that the values of the 3D vectormultiplet fields A_τ and $\widehat{\sigma} = \ell\sigma$ are constant, and they take the following diagonal form.

$$A_\tau = \mathrm{diag}(A_\tau^0, \cdots, A_\tau^{N-1}), \qquad \widehat{\sigma} = \mathrm{diag}(\widehat{\sigma}^0, \cdots, \widehat{\sigma}^{N-1}). \tag{3.3.9}$$

Let $\bar{\mathbf{Z}} \equiv (\bar{Z}_0, \cdots, \bar{Z}_{N-1})^{\mathrm{T}}$ be the first column of g and $\mathbf{Z} \equiv (Z_0, \cdots, Z_{N-1})$ the first row of g^{-1}. The above action can be rewritten as

$$S = m \int \mathrm{d}\tau \, \mathbf{Z}\Big(\frac{\mathrm{d}}{\mathrm{d}\tau} - iA_\tau - \widehat{\sigma}\Big)\bar{\mathbf{Z}}, \qquad |\mathbf{Z}|^2 = 1. \tag{3.3.10}$$

The field $\mathbf{Z}$ transforms as anti-fundamental of the $SU(N)$. One can regard it as the homogeneous coordinate on $\mathbb{CP}^{N-1}$. In terms of $z^I \equiv Z_I/Z_0$ the above action can be further rewritten as

$$S = \int \mathrm{d}\tau \left\{ -i(\vartheta_I D_\tau z^I + \vartheta_{\bar{J}} D_\tau \bar{z}^{\bar{J}}) - \widehat{\sigma}^a P^a \right\}, \tag{3.3.11}$$

from which one can read off the 1-form ϑ, Killing vector X^a and the moment map P^a.

$$\begin{aligned} \vartheta &= \frac{im}{2} \frac{z^I \mathrm{d}\bar{z}^{\bar{I}} - \mathrm{d}z^I \bar{z}^{\bar{I}}}{1 + z^I \bar{z}^{\bar{I}}}, \\ X^{aI}\partial_I &= i\big(T^a_{IJ} z^I + T^a_{0J} - T^a_{I0} z^I z^J - T^a_{00} z^J\big)\partial_J, \\ P^a &= \frac{m}{1 + z^I \bar{z}^{\bar{I}}} \Big\{ (T^a)_{I\bar{J}} z^I \bar{z}^{\bar{J}} + (T^a)_{I0} z^I + (T^a)_{0\bar{J}} \bar{z}^{\bar{J}} + (T^a)_{00} \Big\}. \end{aligned} \tag{3.3.12}$$

Here T^a are $N \times N$ matrices representing the generators of $SU(N)$, and $I, \bar{J} = 1, \cdots, N-1$.

The supersymmetrized theory has N saddle points. One of them corresponds to $\mathbf{Z} = (1, 0, \cdots, 0)$, and the others are all related to it by permutations of the N components. The classical value of the action on this saddle point is $S = -2\pi m(\widehat{\sigma}^0 + iA^0_\tau)$. The localizing Lagrangian near $z^I = \bar{z}^{\bar{I}} = 0$ looks like

$$\begin{aligned} L_{\mathrm{kin}} \simeq m \sum_{I=1}^{N-1} \Big[&\big\{\dot{\bar{z}}^{\bar{I}} - i(A^I_\tau - A^0_\tau)\bar{z}^{\bar{I}}\big\}\big\{\dot{z}^I + i(A^I_\tau - A^0_\tau)z\big\} + (\widehat{\sigma}^I - \widehat{\sigma}^0)^2 \bar{z}^{\bar{I}} z^I \\ &- i\bar{\chi}^{\bar{I}}\big\{\dot{\chi}^I + i(A^I_\tau - A^0_\tau) - (\widehat{\sigma}^I - \widehat{\sigma}^0)\chi^I\big\}\Big]. \end{aligned} \tag{3.3.13}$$

So the contribution to partition function from this saddle point is

$$e^{-S} \prod_{I=1}^{N-1} \frac{\mathrm{Det}\Big[\frac{\mathrm{d}}{\mathrm{d}\tau} + i(A^I_\tau - A^0_\tau) - (\widehat{\sigma}^I - \widehat{\sigma}^0)\Big]}{\mathrm{Det}\Big[\big(\frac{\mathrm{d}}{\mathrm{d}\tau} + i(A^I_\tau - A^0_\tau)\big)^2 - (\widehat{\sigma}^I - \widehat{\sigma}^0)^2\Big]} = \frac{e^{2\pi m \widehat{u}^0}}{\prod_{I=1}^{N-1} 2\sinh \pi(\widehat{u}^0 - \widehat{u}^I)},$$

where $\widehat{u} \equiv \widehat{\sigma} + iA_\tau$. It depends holomorphically on $\widehat{u}$, which is as expected because we started from the action (3.3.8). Summing up the contributions from all saddle points one obtains the full partition function

$$\int \mathcal{D}g e^{-S} = \sum_{w \in \mathcal{W}/\mathcal{W}_K} \frac{e^{2\pi\lambda\cdot w(\widehat{u})}}{\prod_{\alpha\in\Pi^+} 2\sinh \pi\alpha\cdot w(\widehat{u})} = \sum_{I=0}^{N-1} \frac{e^{2\pi m \widehat{u}^I}}{\prod_{J\neq I} 2\sinh(\widehat{u}^I - \widehat{u}^J)}. \tag{3.3.14}$$

This is *not* the character for the M-th symmetric tensor representation of $SU(N)$. One way to fix the mismatch would be to start with the orbit of $\lambda + \tilde{\rho}$ instead of λ, where

$$\tilde{\rho} = \frac{1}{2} \sum_{\alpha\in\Pi^+} \alpha = \frac{1}{2} \sum_{I=1}^{N-1} (\mathbf{e}_0 - \mathbf{e}_I) = \Big(\frac{N-1}{2}, -\frac{1}{2}, \cdots, -\frac{1}{2}\Big). \tag{3.3.15}$$

In other words, replace m by $m + N/2$ at the beginning. Our resolution is not to shift M, but to cancel the anomaly by turning on the Wilson line with "charge" $N/2$.

Chapter 4

GLSM on vortex loops

In this chapter we develop further the description of vortex loops as 1D-3D coupled systems using gauged linear sigma models (GLSMs). These models generally have an independent 1D gauge symmetry in addition to the (global) G symmetry that is gauged by the 3D vectormultiplet. We will see that the Wilson line that cancels the global anomaly for this 1D gauge symmetry naturally resolves the problem of the unwanted shift $\lambda \to \lambda - \tilde{\rho}$.

We begin by reviewing 1D $\mathcal{N}=2$ supersymmetric GLSMs and an exact formula for the Witten indices.

4.1 1D $\mathcal{N}=2$ SUSY GLSMs

A 1D $\mathcal{N}=2$ supersymmetric GLSM consists of a vectormultiplet $(A_t, \sigma, \lambda, \bar{\lambda}, D)$ (3.2.14) for some gauge group G and matter chiral multiplets (ϕ, ψ) and Fermi multiplets (η, F) in some representations of G. The fields in chiral and Fermi multiplets transform under SUSY as

$$\begin{aligned}
\boldsymbol{Q}\phi &= \epsilon\psi, & \boldsymbol{Q}\psi &= \bar{\epsilon}(-iD_t\phi + i\sigma\phi),\\
\boldsymbol{Q}\bar{\phi} &= \bar{\epsilon}\bar{\psi}, & \boldsymbol{Q}\bar{\psi} &= \epsilon(-iD_t\bar{\phi} - i\bar{\phi}\sigma),\\
\boldsymbol{Q}\eta &= \epsilon F + \bar{\epsilon}E, & \boldsymbol{Q}F &= \bar{\epsilon}(-iD_t\eta + i\sigma\eta - \Psi),\\
\boldsymbol{Q}\bar{\eta} &= \bar{\epsilon}\bar{F} + \epsilon\bar{E}, & \boldsymbol{Q}\bar{F} &= \epsilon(-iD_t\bar{\eta} - i\bar{\eta}\sigma - \bar{\Psi}).
\end{aligned} \tag{4.1.1}$$

Here E is a composite field made only of chiral fields of the theory and Ψ is its superpartner. The square of $\boldsymbol{Q}$ acts to all fields as

$$\boldsymbol{Q}^2 = -i\partial_t + i(\sigma + iA_t). \tag{4.1.2}$$

There are various $\boldsymbol{Q}$-invariants which can be used for Lagrangian. First, there are kinetic terms for the three multiplets,

$$\begin{aligned}
L_{\mathrm{v}} &= \mathrm{Tr}\big[(D_t\sigma)^2 - i\bar{\lambda}D_t\lambda + i\bar{\lambda}[\sigma,\lambda] + D^2\big],\\
L_{\mathrm{c}} &= D_t\bar{\phi}D_t\phi - i\bar{\psi}D_t\psi + \bar{\phi}\sigma^2\phi - i\bar{\phi}D\phi - i\bar{\psi}\sigma\psi - i\bar{\phi}\lambda\psi - i\bar{\psi}\bar{\lambda}\phi,\\
L_{\mathrm{f}} &= -i\bar{\eta}D_t\eta + i\bar{\eta}\sigma\eta - \bar{F}F + \bar{E}E - \bar{\eta}\Psi + \bar{\Psi}\eta.
\end{aligned} \tag{4.1.3}$$

Also, supersymmetric interaction terms of chiral multiplets (ϕ_i, ψ_i) and Fermi multiplets $(\eta_i, F_i; E_i)$ can be constructed according to the formula:

$$L_{\text{int}} = \sum_i \left(J_i F_i + \bar{J}_i \bar{F}_i \right) + \sum_{i,j} \left(\psi_j \frac{\partial J_i}{\partial \phi_j} \eta_i + \bar{\psi}_j \frac{\partial \bar{J}_i}{\partial \phi_j} \bar{\eta}_i \right), \tag{4.1.4}$$

where J_i is a composite of chiral fields such that $\sum_i J_i E_i = 0$. This can be regarded as the F-term of the Fermi multiplet with the lowest component (superpotential) $W = \sum_i J_i \eta_i$. In addition, for $U(1)$ vectormultiplets, the Fayet-Iliopoulos term (with coupling ζ) and the Wilson line (with charge q) are also invariant.

$$L_{\text{FI}} = i\zeta D, \qquad L_{\text{WL}} = -q(iA_t + \sigma). \tag{4.1.5}$$

An important role of Wilson lines in 1D GLSMs is to cancel global anomaly. Sometimes Wilson lines with fractional charges become necessary. For example, for a $U(N)$ gauge theory with N_{f} fundamental chirals, N_{a} anti-fundamental chirals, $\widetilde{N}_{\text{f}}$ fundamental Fermis and $\widetilde{N}_{\text{a}}$ anti-fundamental Fermis, the diagonal $U(1)$ subgroup is anomaly free if the Wilson line with the following $U(1)$ charge q is added.

$$q \in -\frac{1}{2}(N_{\text{f}} - N_{\text{a}} + \widetilde{N}_{\text{f}} - \widetilde{N}_{\text{a}}) + \mathbb{Z}. \tag{4.1.6}$$

4.1.1 Witten index.

A powerful formula for the Witten index of 1D $\mathcal{N} = 2$ GLSMs was obtained in [18]. The derivation uses the localization of path integral that follows from the $\boldsymbol{Q}$-exactness of the Lagrangians (4.1.3). The saddle point configurations are can be read from

$$0 = \text{Tr}\left[(D_t\sigma)^2 + D^2 + \cdots \right], \tag{4.1.7}$$

which implies

$$D_t\sigma = D = 0. \tag{4.1.8}$$

At these saddle points, σ and A_t are mutually commuting constants and all other fields must vanish. One can gauge-rotate σ into a Cartan subalgebra $\mathfrak{h} \subset \text{Lie}(G)$, and A_t then takes values in the corresponding maximal torus. The pair (σ, A_t) is further subject to the identification by the action of Weyl group $\mathcal{W}$. The space of saddle points thus becomes a real $2r$-dimensional orbifold, where $r = \text{rk}(G)$. It is useful to define a complex coordinate $u \equiv \sigma + iA_t$ on this space. At this stage, one may also deform the theory by gauging its global symmetry G_{F} by a background vectormultiplet $\widehat{u} \equiv \widehat{\sigma} + i\widehat{A}_t$ satisfying the saddle point condition. Also, for convenience we rescale all the fields and the coordinate t so that the time circle has unit radius.

The index can be obtained by evaluating the one-loop determinant $\Delta(u, \widehat{u})$, multiplying by the Wilson line $e^{-S_{\text{WL}}}$ and then integrating over u. Due to the fact that u is $\boldsymbol{Q}$-closed but $\bar{u}$ is not, the index $I(\widehat{u})$ is expressed as a multiple contour integral of a holomorphic function. Similarly to the discussion in Section 2.2, the one-loop determinant can be expressed in terms of the determinant of $\boldsymbol{Q}^2$. For a quantum mechanics on S^1, the determinant of $\boldsymbol{Q}^2$ is given by products of sinh functions.

$$\operatorname{Det}\boldsymbol{Q}^2 = \operatorname{Det}(-i\partial_t + iu) = \prod_w(-i\partial_t + iw(u)) = \prod_w 2i\sinh\pi w(u). \tag{4.1.9}$$

Thus, the index $I(\widehat{u})$ (up to an overall $\pm$ sign) is given by

$$\begin{aligned} I(\widehat{u}) &= \frac{1}{i^r|\mathcal{W}|}\int \mathrm{d}^r u\, e^{-S_{\mathrm{WL}}(u)}\Delta(u,\widehat{u}), \\ \Delta(u,\widehat{u}) &= \frac{\displaystyle\prod_\alpha 2\sinh\pi(\alpha\cdot u)\prod_i 2\sinh\pi\big(\nu_i\cdot u + \widehat{\nu}_i\cdot\widehat{u}\big)}{\displaystyle\prod_j 2\sinh\pi\big(\mu_j\cdot u + \widehat{\mu}_j\cdot\widehat{u}\big)}. \end{aligned} \tag{4.1.10}$$

Here $(\mu_j,\widehat{\mu}_j)$ runs over the weights of the representation of $G\times G_{\mathrm{F}}$ furnished by chiral multiplets, and similarly $(\nu_i,\widehat{\nu}_i)$ is for the Fermi multiplets.

The contour integral can be performed using the operation called the *Jeffrey-Kirwan (JK) residue*, which means that one only has to collect residue of the poles meeting certain requirement [49, 50]. To simplify the discussion, let us assume that all the poles of Δ are transverse intersection of r singular hyperplanes. Each singular hyperplane is of the form

$$\mu_j\cdot u + \widehat{\mu}_j\cdot\widehat{u} = ik \quad \big(k\in\mathbb{Z}\big),$$

and is labeled by a charge vector $\mu_j\in\mathfrak{h}^*$. Now, the evaluation of JK-residue integral begins by choosing an arbitrary reference charge vector $\eta\in\mathfrak{h}^*$. Then a pole contributes to the integral if η is contained in the cone spanned by the r charge vectors labeling the pole. Note that the set of poles contributing to the integral depends on the choice of η, but the final result of the integral is independent of η.

The function $\Delta(u,\widehat{u})$ has poles in the interior of the space of saddle points as well as at infinity. As was studied in detail in [18] and reviewed in Appendix C, the residue of the pole at infinity may or may not contribute depending on the choice of η as well as the value of the FI coupling ζ. In particular, they do not contribute if η is set equal to ζ, so it is customary to set η as such when studying Witten indices of 1D GLSMs. Note that this implies that the Witten indices do depend on ζ although the FI Lagrangian is $\boldsymbol{Q}$-exact. The GLSMs in general are known to exhibit different behavior depending on the values of ζ, and accordingly the space of FI couplings is divided into several regions or "phases". The index may jump as ζ is varied across phase boundaries. See [18] for more detail.

An example: $\mathbb{CP}^{N-1}$. The GLSM is given by a $U(1)$ gauge theory with N chiral multiplets of charge $+1$ and a positive FI coupling. We turn on the Wilson line with charge q and gauge the flavor $SU(N)$ symmetry by a constant background vectormultiplet $\widehat{u} = \operatorname{diag}(\widehat{u}^0,\cdots,\widehat{u}^{N-1})$. The Witten index is then given by a contour integral

$$I(\widehat{u}) = \int\frac{\mathrm{d}u}{i}\frac{e^{2\pi qu}}{\prod_{J=0}^{N-1}2\sinh\pi(u-\widehat{u}^J)} = \sum_{I=0}^{N-1}\frac{e^{2\pi q\widehat{u}^I}}{\prod_{J\neq I}2\sinh\pi(\widehat{u}^I-\widehat{u}^J)}\,. \tag{4.1.11}$$

The JK-residue integral picks up the contribution of all the N poles $u = \widehat{u}^I$. Without the Wilson line, the integrand is *not* invariant under a large gauge transformation $u \to u + i$ for odd N. This is an example of global anomaly. To obtain the character for the m-th symmetric tensor representation of $SU(N)$ one has to set $q = m + N/2$. We would like to view it as the model with $q = m$ whose anomaly is canceled by the additional Wilson line with $q = N/2$.

4.2 GLSM for vortex worldline quantum mechanics

Let us now turn to the SUSY quantum mechanics on the worldline of vortex loops. We first consider the case where the 3D gauge theory is made of vectormultiplet only. So we take the 3D $\mathcal{N} = 2$ CS theory with $G = SU(N)$ at level k, and put a vortex loop with

$$\begin{aligned}\beta \;&=\; \mathrm{diag}(\beta_1, \cdots, \beta_N) \;=\; \mathrm{diag}\big(\, \underbrace{\beta_{(1)}, \cdots, \beta_{(1)}}_{n_1}, \underbrace{\beta_{(2)}, \cdots, \beta_{(2)}}_{n_2}, \cdots, \underbrace{\beta_{(p)}, \cdots, \beta_{(p)}}_{n_p} \,\big)\,, \\ &\qquad\qquad \beta_{(1)} > \beta_{(2)} > \cdots > \beta_{(p)}\end{aligned} \tag{4.2.1}$$

which breaks G to $K = S[U(n_1) \times \cdots \times U(n_p)]$. For later use let us introduce

$$N_0 = 0, \quad N_1 = n_1, \quad N_2 = n_1 + n_2, \quad \cdots \quad N_p = n_1 + \cdots + n_p = N.$$

The quantum mechanics on the vortex worldline is a 1D $\mathcal{N} = 2$ theory with a global symmetry $G = SU(N)$ which is gauged by the 3D vectormultiplet fields. Also, its Witten index should reproduce the corrected version of (3.2.25):

$$V_\beta(\widehat{\sigma}) = \sum_{w \in \mathcal{W}/\mathcal{W}_K} V^{\mathrm{v}}_\beta(w(\widehat{\sigma})), \qquad V^{\mathrm{v}}_\beta(\widehat{\sigma}) \;=\; \frac{e^{2\pi \sum_i (\lambda+\tilde{\rho})_i \widehat{\sigma}_i}}{\prod_{\beta_i > \beta_j} 2 \sinh \pi(\widehat{\sigma}_i - \widehat{\sigma}_j)}, \tag{4.2.2}$$

where $\mathcal{W}, \mathcal{W}_K, \tilde{\rho}, \Pi^+$ are defined around (3.1.22) and $\lambda, \tilde{\rho}$ are N-component vectors

$$\begin{aligned}&\lambda = (\lambda_1, \cdots, \lambda_N) = \big(\, \underbrace{\lambda_{(1)}, \cdots, \lambda_{(1)}}_{n_1}, \cdots, \underbrace{\lambda_{(p)}, \cdots, \lambda_{(p)}}_{n_p} \,\big)\,, \quad \lambda_{(a)} = k\beta_{(a)}\,, \\ &\tilde{\rho} = (\tilde{\rho}_1, \cdots, \tilde{\rho}_N) = \big(\, \underbrace{\tilde{\rho}_{(1)}, \cdots, \tilde{\rho}_{(1)}}_{n_1}, \cdots, \underbrace{\tilde{\rho}_{(p)}, \cdots, \tilde{\rho}_{(p)}}_{n_p} \,\big)\,, \quad \tilde{\rho}_{(a)} = \frac{1}{2}(N - N_a - N_{a-1})\,.\end{aligned} \tag{4.2.3}$$

By noticing that each $w \in \mathcal{W}/\mathcal{W}_K$ is in one-to-one correspondence with a division of $\{1, \cdots, N\}$ into subsets $d_1, \cdots, d_p$ of order $|d_a| = n_a$, (4.2.2) can also be written as the sum over divisions

$$V_\beta(\widehat{\sigma}) = \sum_{\{d_1, \cdots, d_p\}} \frac{e^{2\pi \sum_i (\lambda+\tilde{\rho})_i \widehat{\sigma}_i}}{\prod_{a<b} \prod_{i \in d_a} \prod_{j \in d_b} 2 \sinh \pi(\widehat{\sigma}_i - \widehat{\sigma}_j)}. \tag{4.2.4}$$

The $V_\beta(\widehat{\sigma})$ in (4.2.2) or (4.2.4) equals the character for the representation of $SU(N)$ with the highest weight λ. The same character formulae work also for $G = U(N)$ by relaxing the tracelessness condition for $\widehat{\sigma}_i$ and modifying the quantization condition for λ_i. To be more

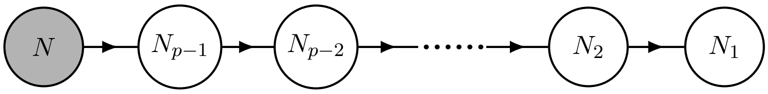

Figure 4.1 The quiver diagram for a GLSM on a vortex loop. It has a 1D $\mathcal{N}=2$ vectormultiplet for each node and a bifundamental chiral multiplet for each solid line connecting the neighboring nodes. The shaded node represents the 3D gauge symmetry.

explicit, recall that we have described the highest weights of $SU(N)$ representations as N-component vectors $\lambda = (\lambda_1, \cdots, \lambda_N)$ satisfying

$$\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0} \quad (i > j), \qquad \sum_{i=1}^{N} \lambda_i = 0.$$

So, λ_i are all equal modulo $\mathbb{Z}$ to m/N for some integer m which gives the charge of the representation under the central subgroup $\mathbb{Z}_N \subset SU(N)$. The highest weight of a $U(N)$ representation is obtained from that of an $SU(N)$ representation λ by a uniform shift of λ_i to make them all integer.

4.2.1 A GLSM and its quiver representation

The GLSM for flag manifolds has been discussed in many places; see [19, 20] for example. Here we study the 1D $\mathcal{N}=2$ version of it. The models can be conveniently described by the quiver diagram of Fig. 4.1. It is a $U(N_{p-1}) \times \cdots \times U(N_1)$ gauge theory with N chiral multiplets in the anti-fundamental of $U(N_{p-1})$ and one bi-fundamental chiral multiplet for each neighboring pair of unitary groups, namely $\mathbf{N}_{a+1} \times \overline{\mathbf{N}}_a$ of $U(N_{a+1}) \times U(N_a)$ for each $a \in \{1, \cdots, p-2\}$. The FI couplings for the diagonal $U(1)^{p-1}$ are chosen to be all negative. In addition, we turn on the following Wilson line for the $U(1)^{p-1}$:

$$L_{\mathrm{WL}} = -\sum_{a=1}^{p-1} q_a \mathrm{Tr}\big(iA_t^{(a)} + \sigma^{(a)}\big), \qquad q_a = k(\beta_{(a)} - \beta_{(a+1)}) + \frac{1}{2}(N_{a+1} - N_{a-1}). \tag{4.2.5}$$

The first term in the formula for q_a is needed so that the model agrees with the adjoint orbit quantization with $\lambda = k\beta/2$. The second term is needed to cancel the global anomaly.

Let us denote the constant value of the $U(N_a)$ vectormultiplet fields at saddle points as

$$\sigma^{(a)} + iA_t^{(a)} = \mathrm{diag}(u_1^{(a)}, \cdots, u_{N_a}^{(a)}).$$

The index is then given by the JK residue integral of the holomorphic function

$$e^{-S_{\mathrm{WL}}(u)}\Delta(u,\widehat{\sigma}) \;=\; \frac{\exp\left(\displaystyle\sum_{a=1}^{p-1}\sum_{i=1}^{N_a} 2\pi q_a u_i^{(a)}\right) \cdot \displaystyle\prod_{a=1}^{p-1}\prod_{i\neq j}^{N_a} 2\sinh\pi(u_i^{(a)} - u_j^{(a)})}{\displaystyle\prod_{i=1}^{N}\prod_{j=1}^{N_{p-1}} 2\sinh\pi(\widehat{\sigma}_i - u_j^{(p-1)}) \prod_{a=1}^{p-2}\prod_{i=1}^{N_{a+1}}\prod_{j=1}^{N_a} 2\sinh\pi(u_i^{(a+1)} - u_j^{(a)})}\,. \tag{4.2.6}$$

At each pole of Δ, the value of the variables $u_i^{(a)}$ are determined one by one through an iterated residue integral. At some of the poles, they are determined according to the following steps. First, each of $u_j^{(p-1)}$ $(j = 1, \cdots, N_{p-1})$ is set equal to one of $\{\widehat{\sigma}_1, \cdots, \widehat{\sigma}_N\}$. Their values must be all different so that the numerator of Δ is nonzero. Once $\{u_j^{(a)}\}_{j=1,\cdots,N_a}$ are determined, then the values of $\{u_j^{(a-1)}\}_{j=1,\cdots,N_{a-1}}$ are chosen in the same way as in the previous step, until all the $u_i^{(a)}$ are determined and a pole is thus specified. Each such pole corresponds to a division of $\{1, \cdots, N\}$ into subsets $d_1, \cdots, d_p$ of order $|d_a| = n_a$. There are $\prod_{a=1}^{p-1} N_a!$ different poles corresponding to the same division, and they all have the same residue. As we will explain shortly, for negative FI couplings these are the only poles which contribute to the JK-residue integral.

The index $I(\widehat{\sigma})$ of the GLSM thus obtained is related to $V_\beta(\widehat{\sigma})$ (4.2.2) as follows:

$$V_\beta(\widehat{\sigma}) \;=\; I(\widehat{\sigma}) \cdot e^{2\pi q \sum_{i=1}^{N} \widehat{\sigma}_i} \;=\; I(\widehat{\sigma}) \cdot W_q(\widehat{\sigma}), \tag{4.2.7}$$

where $q = k\beta_{(p)} - \frac{1}{2}N_{p-1}$. The index reproduces $V_\beta(\widehat{\sigma})$ precisely for $G = SU(N)$. If the 3D gauge group is $G = U(N)$, the GLSM has to be accompanied by a Wilson line of charge q for the diagonal $U(1)$ subgroup of $U(N)$.

4.2.2 Detail of JK-residue integral

Here we explain some detail of the JK-residue integral for our present problem. The basic idea of the JK-residue prescription is presented in Appendix C. Let us denote by $\{\mathbf{e}_i^{(a)}\}_{i=1,\cdots,N_a}^{a=1,\cdots,p-1}$ the basis vectors for the space of charges. The singular hyperplanes of Δ (4.2.6) are then labeled by the charge vectors of the form

$$\mathbf{q}_j \equiv -\mathbf{e}_j^{(p-1)} \qquad \text{or} \qquad \mathbf{q}_{ij}^{(a)} \equiv \mathbf{e}_i^{(a+1)} - \mathbf{e}_j^{(a)}. \tag{4.2.8}$$

The dimension of the space of charges is $r = \sum_{a=1}^{p-1} N_a$.

At each pole, the values of $u_i^{(a)}$ are determined one by one through an iterated residue integral. The process can be regarded as if the u-variables are connected together into some trees each starting at one of the $\widehat{\sigma}_i$'s. At the same time, a set Π of r charge vectors are chosen from (4.2.8), and all the basis vectors $\mathbf{e}_i^{(a)}$ are expressed as their linear combinations. As an example, take $N = 4, (N_3, N_2, N_1) = (3, 2, 1)$ and consider a pole

$$\begin{aligned} \widehat{\sigma}_1 &= u_1^{(3)}, \\ \widehat{\sigma}_2 &= u_2^{(3)} \;=\; u_1^{(2)} \;=\; u_1^{(1)}, \\ \widehat{\sigma}_4 &= u_3^{(3)} \;=\; u_2^{(2)}. \end{aligned} \tag{4.2.9}$$

Then all the basis vectors $\mathbf{e}_i^{(a)}$ are expressed as linear combinations of the 6 charge vectors in $\Pi = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_{21}^{(2)}, \mathbf{q}_{32}^{(2)}, \mathbf{q}_{11}^{(1)}\}$:

$$\begin{aligned} &\mathbf{e}_1^{(3)} = -\mathbf{q}_1, \\ &\mathbf{e}_2^{(3)} = -\mathbf{q}_2, \qquad \mathbf{e}_1^{(2)} = -\mathbf{q}_2 - \mathbf{q}_{21}^{(2)}, \qquad \mathbf{e}_1^{(1)} = -\mathbf{q}_2 - \mathbf{q}_{21}^{(2)} - \mathbf{q}_{11}^{(1)}, \\ &\mathbf{e}_3^{(3)} = -\mathbf{q}_3, \qquad \mathbf{e}_2^{(2)} = -\mathbf{q}_3 - \mathbf{q}_{32}^{(2)}. \end{aligned} \tag{4.2.10}$$

The form of the trees can be read from (4.2.9), and the relations (4.2.10) indicate how each $u_i^{(a)}$ is connected to one of the $\widehat{\sigma}_i$'s by a unique path along the trees. In the above example, the elements of Π always appear in the RHS of the relations (4.2.10) with *negative* coefficients because the trees have grown only in the decreasing direction of a. Since the reference charge vector η (C.2.5) is given by

$$\eta = \sum_{a=1}^{p-1} \zeta_a \sum_{i=1}^{N_a} \mathbf{e}_i^{(a)}, \qquad (\zeta_a < 0) \tag{4.2.11}$$

it is a *positive* linear combination of the elements of Π, and therefore the pole (4.2.9) contributes to the JK-residue integral. The same argument applies to all the poles described in the paragraph after (4.2.6): all of them contribute to the index since the corresponding trees extend only in the decreasing direction of a.

In fact, Δ (4.2.6) has other poles corresponding to (i) trees with branchings or (ii) trees part of which grow in the wrong direction. Both types of the poles have vanishing residues, but those of type (ii) are also excluded by the rule of JK-residue. If parts of the trees grow in the wrong direction, some elements of Π appear in the expression for η with wrong sign.

4.2.3 Another GLSM.

There is another GLSM whose Witten index reproduces $V_\beta(\widehat{\sigma})$ (4.2.2) up to sign. It has the gauge group $U(\tilde{N}_{p-1}) \times \cdots \times U(\tilde{N}_1)$, where $\tilde{N}_a = N - N_a$, with one bifundamental chiral for each neighboring pair of unitary groups and N chirals in the fundamental of $U(\tilde{N}_1)$. The model is described by the quiver diagram of Section 4.2.3. The $p-1$ FI couplings are all chosen to be positive. In addition, we need Wilson line with the $U(1)^{p-1}$ charge

$$\tilde{q}_a = k(\beta_{(a+1)} - \beta_{(a)}) + \frac{1}{2}(\tilde{N}_{a+1} - \tilde{N}_{a-1}). \tag{4.2.12}$$

The index for this GLSM can be computed in the same way as in the previous model. It satisfies (4.2.7) with $q = k\beta_{(1)} + \frac{1}{2}\tilde{N}_1$.

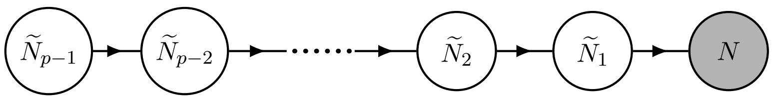

Figure 4.2 The quiver diagram of another GLSM for the same flag manifold.

4.2.4 More alternatives.

In addition to the two quiver theories presented above, there are two series of alternative quiver theory realizations for the sigma model whose target is the same flag manifold. The first is defined by the quiver diagram of Fig. 4.3. In addition to the bifundamental chiral multiplets for neighboring pairs of nodes, the theory has a Fermi multiplet in $\mathbf{N}_{s-1} \times \overline{\tilde{\mathbf{N}}}_s$ of $U(N_{s-1}) \times U(\tilde{N}_s)$.

The FI couplings for the $U(N_a)$ are all negative while those for $U(\tilde{N}_a)$ are all positive. In addition, we need the Wilson line with $U(1)^{p-1}$ charge

$$
\begin{aligned}
q_a &= k(\beta_{(a)} - \beta_{(a+1)}) + \frac{1}{2}(N_{a+1} - N_{a-1}), \qquad (a = 1, \cdots, s-1) \\
\tilde{q}_a &= k(\beta_{(a+1)} - \beta_{(a)}) + \frac{1}{2}(\tilde{N}_{a+1} - \tilde{N}_{a-1}). \qquad (a = s, \cdots, p-1)
\end{aligned}
\tag{4.2.13}
$$

The index of the model satisfies (4.2.7) with $q = k\beta_{(s)} + \frac{1}{2}(\tilde{N}_s - N_{s-1})$. This series interpolates the previous two GLSM descriptions.

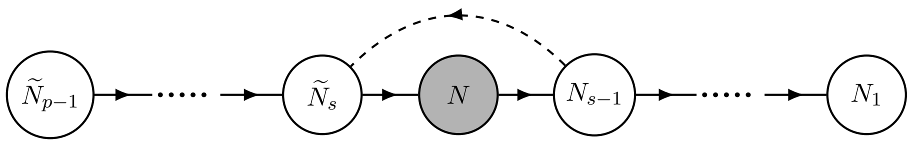

Figure 4.3 The quiver diagram describing a series of GLSMs for the same flag manifold. The dashed line represents a bifundamental Fermi multiplet.

The second series of GLSMs is defined by the quiver diagrams of Fig. 4.4 which have one more node than the previous ones. The FI couplings for the $U(N_a)$ are all negative while those for $U(\tilde{N}_a)$ are all positive. In addition, we need the Wilson line with the following $U(1)^p$ charges:

$$
\begin{aligned}
q_a &= k(\beta_{(a)} - \beta_{(a+1)}) + \frac{1}{2}(N_{a+1} - N_{a-1}) \ (a < s), \quad q_s = k(\beta_{(s)} - \beta_*) + \frac{1}{2}(N - N_{s-1} - \tilde{N}_s), \\
\tilde{q}_a &= k(\beta_{(a+1)} - \beta_{(a)}) + \frac{1}{2}(\tilde{N}_{a+1} - \tilde{N}_{a-1}) \ (a > s), \quad \tilde{q}_s = k(\beta_{(s+1)} - \beta_*) + \frac{1}{2}(\tilde{N}_{s+1} + N_s - N),
\end{aligned}
\tag{4.2.14}
$$

where β_* is a parameter which is constrained only by the anomaly cancellation condition. The index of this model satisfies (4.2.7) with $q = k\beta_* + \frac{1}{2}(\tilde{N}_s - N_s)$. Note that the second series for β (4.2.1) can be thought of as the first series for

$$
\beta = \mathrm{diag}\big(\underbrace{\beta_{(1)}, \cdots, \beta_{(1)}}_{n_1}, \cdots, \underbrace{\beta_{(s)}, \cdots, \beta_{(s)}}_{n_s}, \underbrace{\beta_*, \cdots, \beta_*}_{0}, \underbrace{\beta_{(s+1)}, \cdots, \beta_{(s+1)}}_{n_{s+1}}, \cdots, \underbrace{\beta_{(p)}, \cdots, \beta_{(p)}}_{n_p}\big).
$$

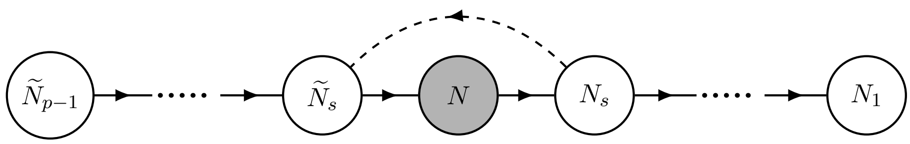

Figure 4.4 Quivers for another series of GLSMs for the same flag manifold.

It is tempting to identify $\beta_{(a)}$'s as some kind of position coordinates. The formulae for $q_a, \tilde{q}_a$ suggest that the a-th gauge node (white node) corresponds to *branes* stretching between $\beta = \beta_{(a)}$ and $\beta = \beta_{(a+1)}$. One might also think that the position of the 3D gauge node (shaded node) should be determined according to the value of $\beta_{(a)}$'s, but this is not the case. The bulk 3D $U(N)$ gauge theory has fields in the adjoint representation only, so there are no fields charged under the diagonal $U(1)$ subgroup of $U(N)$. The expectation value of a vortex loop should therefore be invariant under the uniform shift $\beta_{(a)} \to \beta_{(a)} + c$. As we will see in the next section, the situation changes when matters in (anti-)fundamental representation of $U(N)$ are introduced.

Let us briefly explain how these alternatives give the same flag manifolds as the space of classical vacua, by taking the example for the Grassmannian $\mathrm{Gr}(n,N) = U(N)/\big(U(n)\times U(N-n)\big)$. The usual model is the $U(n)$ gauge theory with N fundamental chiral multiplets A_{iI} ($i = 1,\cdots,n, I = 1,\cdots,N$), as described by the quiver diagram of Fig. 4.5 left. The classical vacuum equation is

$$\sum_I A_{iI}\bar{A}_{Ij} = \zeta\delta_{ij},$$

where ζ is the FI coupling for the diagonal $U(1)$ subgroup of $U(n)$. For $\zeta > 0$, each solution gives a set of n orthonormal N-component complex vectors. The equivalence classes of solutions with respect to $U(n)$ define n-dimensional hyperplanes in $\mathbb{C}^N$, and the space of such hyperplanes is $\mathrm{Gr}(n,N)$.

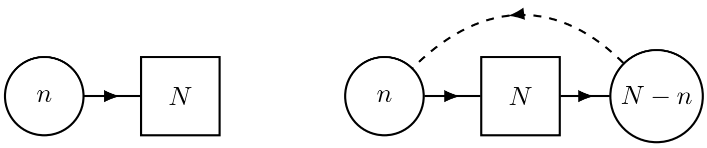

Figure 4.5 The usual and alternative quivers of the GLSM for the Grassmannian $\mathrm{Gr}(n,N)$.

The alternative model is the $U(n)\times U(N-n)$ gauge theory with N chiral multiplets in the fundamental of $U(n)$, N chiral multiplets in the anti-fundamental of $U(N-n)$ and a Fermi multiplet in the bifundamental of $U(N-n)\times U(n)$ as described by the quiver of Fig. 4.5 right. Let us denote the bottom components of these multiplets as

$$A_{iI},\quad B_{I\tilde{\jmath}},\quad \eta_{\tilde{\jmath}i} \qquad (i=1,\cdots,n\,;\ \tilde{\jmath}=1,\cdots,N-n\,;\ I=1,\cdots,N).$$

In the presence of the superpotential $W = \sum_{i,\tilde{\jmath},I} A_{iI}B_{I\tilde{\jmath}}\eta_{\tilde{\jmath}i}$, the classical vacuum equations are

$$\sum_I A_{iI}\bar{A}_{Ij} = \zeta\delta_{ij}, \quad \sum_I \bar{B}_{\tilde{\imath}I}B_{I\tilde{\jmath}} = -\tilde{\zeta}\delta_{\tilde{\imath}\tilde{\jmath}}, \quad A_{iI}B_{I\tilde{\jmath}} = 0,$$

where $\zeta,\tilde{\zeta}$ are the FI couplings for $U(n)$ and $U(N-n)$. If $\zeta > 0$ and $\tilde{\zeta} < 0$, each solution of these equations defines an n-plane and a $(N-n)$-plane in $\mathbb{C}^N$ that are orthogonal to each other. The space of such pairs is again given by $\mathrm{Gr}(n,N)$.

In view of the fact that many alternative GLSMs presented in this section give the same Witten index and vacuum manifold, we suspect they are all dual to one another.

4.3 Theories with matters

Here we study vortex loops in 3D $U(N)$ gauge theories with various matter chiral multiplets. The path integral with respect to the added chiral multiplets on the vortex background modifies $V_\beta(\widehat{\sigma})$ (4.2.2) according to the formula in Section 2.1. We would like to find the corresponding modification of the quiver GLSMs introduced in the last section.

4.3.1 Global symmetry of the 1D theory

For a vortex loop in a theory with chiral multiplets of real mass m and R-charge r, the function $V_\beta(\widehat{\sigma})$ will also depend on m, r and the squashing parameter b. Since m is in a 3D vectormultiplet, $\widehat{m} \equiv \ell m$ appears in the 1D theory on the vortex worldline according to the same rule as that for $\widehat{\sigma}$. In fact, the other parameters r, b also appear in the 1D theory through the background gauging of a specific global $U(1)$ symmetry.

The 3D $\mathcal{N} = 2$ theory on an ellipsoid has the translation symmetry $U(1)_\tau \times U(1)_\varphi$ and the R-symmetry $U(1)_{\mathrm{R(3D)}}$. The $U(1)_\tau$ descends to the translation symmetry along the vortex loop, whereas $U(1)_\varphi$ appears in the 1D theory as a global symmetry. The R-symmetry of the 1D $\mathcal{N} = 2$ SUSY theory should be a linear combination of $U(1)_\varphi$ and $U(1)_{\mathrm{R(3D)}}$ (and other abelian global symmetries if there are any). However, the Witten index is independent of the assignment of this R-charge on matters because the square of the 1D SUSY (3.2.14), (4.1.1) does not contain the R-symmetry. But the index does depend on the charge assignments of the other non-R linear combination of $U(1)_\varphi$ and $U(1)_{\mathrm{R(3D)}}$, as we now explain.

The SUSY of the 3D theory on an ellipsoid squares to

$$\boldsymbol{Q}^2_{(3\mathrm{D})} = \frac{1}{\ell}\mathrm{H} + \frac{1}{\tilde{\ell}}\mathrm{M} - \frac{1}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\right)\mathrm{R}_{(3\mathrm{D})} + i\Big(\sigma + \frac{i}{\ell}A_\tau + \frac{i}{\tilde{\ell}}A_\varphi\Big) + im,$$

where H and M are operators that act on dynamical fields as $-i\mathcal{L}_{\partial_\tau}$ and $-i\mathcal{L}_{\partial_\varphi}$, respectively. In section 3.2.2 we have made contact of this $\boldsymbol{Q}_{(3\mathrm{D})}$ with the 1D SUSY on the vortex worldline using the fact that the cohomological variables transform under $\boldsymbol{Q}_{(3\mathrm{D})}$ like 1D $\mathcal{N} = 2$ multiplets. So, let us study the action of $\boldsymbol{Q}^2_{(3\mathrm{D})}$ on cohomological variables on top of the vortex worldline. As an example take $\Psi = \xi\psi$ (2.2.4) which is the superpartner of a chiral scalar ϕ. With the understanding that $\mathrm{H}, \mathrm{M}, \mathrm{R}_{(3\mathrm{D})}$ act only on a dynamical field ψ and not ξ, one finds

$$\begin{aligned} \boldsymbol{Q}^2_{(3\mathrm{D})}\Psi\Big|_{\theta=0} &= \xi \cdot \left\{\frac{1}{\ell}\mathrm{H} + \frac{1}{\tilde{\ell}}\mathrm{M} - \frac{1}{2}\left(\frac{1}{\ell} + \frac{1}{\tilde{\ell}}\right)\mathrm{R}_{(3\mathrm{D})} + i\Big(\sigma + \frac{i}{\ell}A_\tau\Big) + im\right\}\psi \\ &= \frac{i}{\ell}\left\{-\partial_\tau + \frac{ibQ}{2}\big(\mathrm{R}_{(3\mathrm{D})} - 2\mathrm{M}\big) + (\widehat{\sigma} + iA_\tau) + \widehat{m}\right\}\Psi. \end{aligned} \tag{4.3.1}$$

Here we used $\mathcal{L}_{\partial_\tau}\xi = \mathcal{L}_{\partial_\varphi}\xi = \frac{i}{2}\xi$ and also that $\mathcal{L}_{\partial_\varphi}\Psi = 0$ along the vortex worldline because Ψ is a Lorentz scalar. The above computation works for all the cohomological variables. Thus the SUSY squared of the vortex worldline theory should take the form (here t is the worldline coordinate of period 2π):

$$
\begin{aligned}
\boldsymbol{Q}^2_{(1\mathrm{D})} &\sim \; -\partial_t + \frac{ibQ}{2}\mathrm{G} + (\widehat{\sigma} + iA_\tau) + \widehat{m} + (\text{1D vectormultiplet fields})\,, \\
\mathrm{G} &\equiv \mathrm{R}_{(3\mathrm{D})} - 2\mathrm{M}\,.
\end{aligned}
\tag{4.3.2}
$$

Note that G is a non-R global symmetry in the sense of both 3D and 1D. The interpretation of the second term in the RHS is that the global $U(1)_{\mathrm{G}}$ symmetry of the vortex worldline theory is gauged by the background field:

$$
\sigma^{\mathrm{G}} + iA_t^{\mathrm{G}} = \frac{ibQ}{2}. \tag{4.3.3}
$$

4.3.2 Adjoint representation

Let us first consider the case with an adjoint chiral multiplet with mass m and R-charge r. According to the result of Section 2.1, the function V_β now consists of the contribution from vector and chiral multiplets:

$$
V_\beta(\widehat{\sigma}) = \sum_{w\in\mathcal{W}/\mathcal{W}_K} V_\beta^{\mathrm{v}}(w(\widehat{\sigma}))V_\beta^{\mathrm{c}}(w(\widehat{\sigma})). \tag{4.3.4}
$$

Here $V_\beta^{\mathrm{v}}(\widehat{\sigma})$ is given in (4.2.2) and

$$
\begin{aligned}
V_\beta^{\mathrm{c}\mathbf{1}}(\widehat{\sigma}) &= \prod_{\beta_i>\beta_j} 2\sinh\pi\Big(\widehat{\sigma}_i - \widehat{\sigma}_j - \widehat{m} - \frac{irbQ}{2}\Big), \\
V_\beta^{\mathrm{c}\mathbf{2}}(\widehat{\sigma}) &= \prod_{\beta_i>\beta_j} \Big(2\sinh\pi\Big(\widehat{\sigma}_i - \widehat{\sigma}_j + \widehat{m} - \frac{i(2-r)bQ}{2}\Big)\Big)^{-1},
\end{aligned}
\tag{4.3.5}
$$

depending on the choice of boundary condition **BC1** or **BC2**. Suitable 1D $\mathcal{N}=2$ SUSY theories should reproduce these as the Witten index up to a freedom of additional Wilson lines. It is natural to expect that such theories can be obtained by modifying the GLSMs introduced in the previous section. We take the theory of Fig. 4.3 as the starting point.

BC1. Let us consider a GLSM corresponding to the quiver diagram of Fig. 4.6 which is obtained by adding links to the quiver of Fig. 4.3. The matters corresponding to the added links are charged under $U(1)_{\mathrm{G}}$ as well as $U(1)_{\mathrm{m}}$ corresponding to the 3D real mass. We denote their generators by G and m.

The matter multiplets and their charges are as follows. Each gauge group has an adjoint chiral multiplet with $\mathrm{m}=1$ and $\mathrm{G}=r$. Each pair of neighboring nodes has a bifundamental chiral multiplet with $\mathrm{m}=\mathrm{G}=0$ and a bifundamental Fermi multiplet with $\mathrm{m}=1, \mathrm{G}=r$. In addition, there is a Fermi multiplet with $\mathrm{m}=\mathrm{G}=0$ and a chiral multiplet with $\mathrm{m}=-1, \mathrm{G}=-r$ in the bifundamental of $U(N_{s-1})\times U(\tilde{N}_s)$. The FI couplings are negative for $U(N_a)$ and positive for $U(\tilde{N}_a)$ gauge groups.

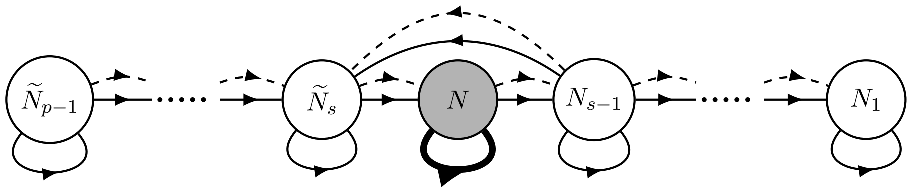

Figure 4.6 The worldline theory for a vortex loop in 3D $U(N)$ gauge theory with an adjoint chiral multiplet (represented by a thick line) satisfying **BC1**.

The theory is free of global anomaly, so the charge of Wilson line is determined by the CS coupling and β only.

$$\begin{aligned} q_a &= k(\beta_{(a)} - \beta_{(a+1)}), \qquad (a = 1, \cdots, s-1) \\ \tilde{q}_a &= k(\beta_{(a+1)} - \beta_{(a)}). \qquad (a = s, \cdots, p-1) \end{aligned} \tag{4.3.6}$$

However, when $V^{\mathrm{v}}_{\beta}(\widehat{\sigma})$ was re-defined in (4.2.2), we also included the Wilson line factor $e^{2\pi \sum_i \tilde{\rho}_i \widehat{\sigma}_i}$ which cancels the global anomaly of the quiver theory for Fig. 4.3. The added massive 1D matters bring about another global anomaly, but it can be canceled by a Wilson line factor $e^{-2\pi \sum_i \tilde{\rho}_i \widehat{\sigma}_i}$. Thus $V^{\mathrm{c}\mathbf{1}}_{\beta}(\widehat{\sigma})$ needs to be corrected by this Wilson line factor.

The Witten index is the JK-residue integral of the following one-loop determinant Δ multiplied by the Wilson line with charges (4.3.6):

$$\begin{aligned} \Delta \;=\;& (2\sinh \pi \widetilde{m})^{-\sum_a N_a - \sum_a \tilde{N}_a} \\ &\times \prod_{a=1}^{s-1} \prod_{i\neq j}^{N_a} \frac{2\sinh \pi(u_i^{(a)} - u_j^{(a)})}{2\sinh \pi(u_i^{(a)} - u_j^{(a)} + \widetilde{m})} \prod_{a=s}^{p-1} \prod_{i\neq j}^{\tilde{N}_a} \frac{2\sinh \pi(\tilde{u}_i^{(a)} - \tilde{u}_j^{(a)})}{2\sinh \pi(\tilde{u}_i^{(a)} - \tilde{u}_j^{(a)} + \widetilde{m})} \\ &\times \prod_{a=s}^{p-2} \prod_{i=1}^{\tilde{N}_{a+1}} \prod_{j=1}^{\tilde{N}_a} \frac{2\sinh \pi(\tilde{u}_i^{(a+1)} - \tilde{u}_j^{(a)} + \widetilde{m})}{2\sinh \pi(\tilde{u}_i^{(a+1)} - \tilde{u}_j^{(a)})} \cdot \prod_{i=1}^{\tilde{N}_s} \prod_{j=1}^{N} \frac{2\sinh \pi(\tilde{u}_i^{(s)} - \widehat{\sigma}_j + \widetilde{m})}{2\sinh \pi(\tilde{u}_i^{(s)} - \widehat{\sigma}_j)} \\ &\times \prod_{i=1}^{N} \prod_{j=1}^{N_{s-1}} \frac{2\sinh \pi(\widehat{\sigma}_i - u_j^{(s-1)} + \widetilde{m})}{2\sinh \pi(\widehat{\sigma}_i - u_j^{(s-1)})} \cdot \prod_{a=1}^{s-2} \prod_{i=1}^{N_{a+1}} \prod_{j=1}^{N_a} \frac{2\sinh \pi(u_i^{(a+1)} - u_j^{(a)} + \widetilde{m})}{2\sinh \pi(u_i^{(a+1)} - u_j^{(a)})} \\ &\times \prod_{i=1}^{N_{s-1}} \prod_{j=1}^{\tilde{N}_s} \frac{2\sinh(u_i^{(s-1)} - \tilde{u}_j^{(s)})}{2\sinh(u_i^{(s-1)} - \tilde{u}_j^{(s)} - \widetilde{m})}, \end{aligned} \tag{4.3.7}$$

where we used $\widetilde{m} \equiv \widehat{m} + \frac{ibQr}{2}$. In the limit $\widehat{m} \to -\infty$ the one-loop determinants for the massive multiplets turn into Wilson lines. The above Δ then reduces to

$$\exp\left(-\pi \widetilde{m} \sum_{a<b} n_a n_b\right) \cdot W_{\frac{1}{2}(\tilde{N}_s - N_{s-1})}(\widehat{\sigma})$$

times the Δ for the quiver GLSM of Fig. 4.3 and a Wilson line factor that shift the charges (4.3.6) back to (4.2.13). On the other hand, we will see in Chapter 5 that the 1D theory has an enhanced SUSY when $\widehat{m} = 0$.

The above one-loop determinant Δ has more poles than the one corresponding to Fig. 4.3 due to the added chiral multiplets. However, as we explain in the next paragraph, none of those new poles contribute to the index according to the rule of JK-residue. This is in accordance with the fact that $V_\beta(\widehat{\sigma})$ (4.3.4) is given by a sum over elements of $\mathcal{W}/\mathcal{W}_K$ as in pure CS theory. Once one accepts this fact, it is straightforward to check that the index reproduces (4.3.4) for **BC1**.

Detail of JK-residue integral (2). Here we discuss some detail of the JK-residue integral with the above Δ in the integrand. The space of charges is of dimension $r = \sum_{a=1}^{s-1} N_a + \sum_{a=s}^{p-1} \tilde{N}_a$ and we denote its basis vectors by $\{\mathbf{e}_i^{(a)}\}_{i=1,\cdots,N_a}^{a=1,\cdots,s-1}$, $\{\tilde{\mathbf{e}}_i^{(a)}\}_{i=1,\cdots,\tilde{N}_a}^{a=s,\cdots,p-1}$. Let us first list the charge vectors labeling the singular hyperplanes of Δ. The hyperplanes which are present before introducing the 3D adjoint chiral multiplet are labeled by the charges

$$\tilde{\mathbf{q}}_{ij}^{(a)} \equiv \tilde{\mathbf{e}}_i^{(a)} - \tilde{\mathbf{e}}_j^{(a-1)}, \qquad \tilde{\mathbf{q}}_i \equiv \tilde{\mathbf{e}}_i^{(s)}, \qquad \mathbf{q}_i \equiv -\mathbf{e}_i^{(s-1)}, \qquad \mathbf{q}_{ij}^{(a)} \equiv \mathbf{e}_i^{(a+1)} - \mathbf{e}_j^{(a)}.$$

The hyperplanes corresponding to the added chiral multiplets are labeled by

$$\mathbf{p}_{ij}^{(a)} \equiv \mathbf{e}_i^{(a)} - \mathbf{e}_j^{(a)} + \mathbf{m}, \qquad \tilde{\mathbf{p}}_{ij}^{(a)} \equiv \tilde{\mathbf{e}}_i^{(a)} - \tilde{\mathbf{e}}_j^{(a)} + \mathbf{m}, \qquad \mathbf{r}_{ij} \equiv \mathbf{e}_j^{(s-1)} - \tilde{\mathbf{e}}_i^{(s)} - \mathbf{m},$$

where we included the generator $\mathbf{m}$ of the $U(1)_{\mathrm{m}}$ for convenience.

As in the previous example, the iterative residue integral at each pole determines the values of the variables $u_j^{(a)}, \tilde{u}_j^{(a)}$ one by one. The process can be viewed as if those variables are linked together to form trees each starting from one of the $\widehat{\sigma}_i$. At the same time, the process also picks up from the above list a set Π of r charge vectors that play the role of the links. All the basis vectors $\mathbf{e}_i^{(a)}, \tilde{\mathbf{e}}_i^{(a)}$ are then expressed as linear combinations of the elements of Π. Now, to decide whether the pole contributes to the JK-residue integral, one expresses the reference charge vector (C.2.5)

$$\eta \;=\; \sum_{a=1}^{s-1}\sum_{i=1}^{N_a} \zeta_a \mathbf{e}_i^{(a)} + \sum_{a=s}^{p-1}\sum_{i=1}^{\tilde{N}_a} \tilde{\zeta}_a \tilde{\mathbf{e}}_i^{(a)} \qquad (\zeta_a < 0, \quad \tilde{\zeta}_a > 0)$$

as a linear combination of the elements of Π, and checks if the coefficients are all positive. As we observed in the previous simpler example, the sign of the coefficient of a given element of Π is to a large extent related to the direction in which the trees grow at the corresponding link.

There are a few conditions that a pole must satisfy in order to contribute to the integral. One can prove them step by step. First, $\mathbf{r}_{ij}$ cannot participate in Π. Then, all the basis vectors $-\mathbf{e}_i^{(a)}$ must be expressed as non-negative linear combinations of $\{\mathbf{q}_j, \mathbf{q}_{jk}^{(b)}, \mathbf{p}_{jk}^{(b)}\}$, and similarly all $\tilde{\mathbf{e}}_i^{(a)}$ must be non-negative linear combinations of $\{\tilde{\mathbf{q}}_j, \tilde{\mathbf{q}}_{jk}^{(b)}, \tilde{\mathbf{p}}_{jk}^{(b)}\}$. In terms of the formation of trees these conditions can be phrased as follows: each tree consists of u-variables only or $\tilde{u}$-variables only. A tree of u-variables can only be extended by attaching a new variable $u_j^{(b)}$ according to

$$\widehat{\sigma}_j = u_j^{(b)} \;\; (\text{for } b = s-1) \qquad \text{or} \qquad u_k^{(b+1)} = u_j^{(b)} \qquad \text{or} \qquad u_k^{(b)} + \tilde{m} = u_j^{(b)}.$$

Likewise, a tree of $\tilde{u}$-variables can only be extended by attaching $\tilde{u}_j^{(b)}$ according to

$$\tilde{u}_j^{(b)} = \widehat{\sigma}_j \quad (\text{for } b = s) \qquad \text{or} \qquad \tilde{u}_j^{(b)} = \tilde{u}_k^{(b-1)} \qquad \text{or} \qquad \tilde{u}_j^{(b)} = \tilde{u}_k^{(b)} - \tilde{m}.$$

For each pole satisfying the above conditions we study whether the residue is nonvanishing. In fact, due to the determinants of vector and Fermi multiplets in the numerator of Δ, the residue vanishes if Π contains $\mathbf{p}_{ij}^{(a)}$ or $\tilde{\mathbf{p}}_{ij}^{(a)}$. The residue also vanishes when two or more trees start from a single $\widehat{\sigma}_i$, or when there are trees with branchings. Thus the trees must consist only of (i) linear chains of u-variables extending in the decreasing direction of a and (ii) linear chains of $\tilde{u}$-variables extending in the increasing direction of a. Moreover, each $\widehat{\sigma}_i$ can have at most one chain starting from it. The set of poles contributing to the JK-residue integral is therefore the same as before introducing the adjoint chiral multiplet in 3D, and it is precisely what is needed for the integral to reproduce (4.3.4).

BC2. For this boundary condition, the GLSM on the vortex worldline is described by the quiver diagram of Fig. 4.7 which has extra links compared to the quiver of Fig. 4.3.

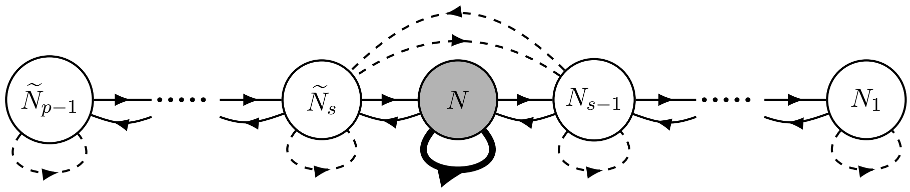

Figure 4.7 The quiver GLSM on the vortex worldline for 3D $U(N)$ gauge theory with an adjoint chiral multiplet satisfying **BC2**.

The matter content and the charge assignment are as follows. For each $U(N_a)$ or $U(\tilde{N}_a)$ gauge node, it has a vectormultiplet as well as an adjoint Fermi multiplet with $\mathrm{m} = -1$ and $\mathrm{G} = 2-r$. Each pair of neighboring nodes has a bifundamental and an anti-bifundamental chiral multiplets, and the latter has $\mathrm{m} = +1, \mathrm{G} = r-2$. In addition, the pair $U(N_{s-1}) \times U(\tilde{N}_s)$ has one bifundamental and one anti-bifundamental Fermi multiplets, the latter carrying $\mathrm{m} = -1$ and $\mathrm{G} = 2 - r$. The FI couplings for $U(N_a)$ are all negative while those for $U(\tilde{N}_a)$ are all positive. As in the previous case of **BC1**, the model is free of global anomaly. The charge of the Wilson line can be chosen the same way as (4.3.6), and the function $V_\beta^{c\mathbf{2}}(\widehat{\sigma})$ needs to be corrected by a Wilson line factor $e^{-2\pi \sum_i \tilde{\rho}_i \widehat{\sigma}_i}$. We will not go into the detail of the JK-residue evaluation as it is somewhat simpler than the previous case.

In the limit $\widehat{m} \to -\infty$ the massive matters turn into a Wilson line of appropriate $U(1)^{p-1}$ charge and the model reduces to that for the quiver of Fig. 4.3. On the other hand, the 1D field content is such that the supersymmetry enhances to $\mathcal{N} = 4$ if $\widehat{m}$ is turned off and an appropriate superpotential interaction is turned on. The m, G-charges of the adjoint Fermi multiplets were chosen so that the superpotential terms are invariant. However, the enhanced $\mathcal{N} = 4$ SUSY here is qualitatively different from the one for **BC1**: they have different kind of multiplets and

R-symmetries. Also, as we will see in the next section, the SUSY enhancement here does not seem to be related to the enhancement of bulk 3D SUSY.

4.3.3 Fundamental representation

Next we consider vortex loops in 3D $U(N)$ gauge theory with a fundamental chiral multiplet of mass m and R-charge r. We regard that the matter is in a bifundamental of $U(N) \times U(1)_{\rm m}$. According to the result of section 2.1, the function $V_\beta(\widehat{\sigma})$ is given by (4.3.4) with

$$\begin{aligned} V_\beta^{\rm c\mathbf{1}}(\widehat{\sigma}) &= \prod_{\beta_i<0} 2\sinh\pi\Big(\widehat{\sigma}_i - \widehat{m} + \frac{irbQ}{2}\Big), \\ V_\beta^{\rm c\mathbf{2}}(\widehat{\sigma}) &= \prod_{\beta_i>0} \Big(2\sinh\pi\Big(\widehat{\sigma}_i - \widehat{m} - \frac{i(2-r)bQ}{2}\Big)\Big)^{-1}, \end{aligned} \tag{4.3.8}$$

depending on the choice of boundary condition. But a simple multiplication of these products of sinh functions will lead to a global anomaly, so we also need a suitable Wilson line. It is also known that the introduction of (anti-)fundamental chiral multiplets shifts the effective CS and FI couplings [51].

As in the previous subsection, we construct the vortex worldline theory as a modification of the quiver GLSM of Fig. 4.3. Let us also assume

$$\beta_{(s+1)} < \beta_{(s)} = 0 < \beta_{(s-1)}. \tag{4.3.9}$$

Then it turns out that the necessary modification of the quiver is to add just one link connecting a 1D gauge node and the flavor $U(1)_{\rm m}$ node as shown in Fig. 4.8. Depending on the choice of boundary condition, we introduce

(**BC1**) a Fermi multiplet in the bifundamental of $U(1)_{\rm m} \times U(\tilde{N}_s)$ with $\mathrm{G} = -r$,

(**BC2**) a chiral multiplet in the bifundamental of $U(N_{s-1}) \times U(1)_{\rm m}$ with $\mathrm{G} = r-2$.

The added links reproduce precisely the contribution of the 3D fundamental chiral multiplet to $V_\beta(\widehat{\sigma})$ (4.3.8), but the 1D theory now has global anomaly. It can be canceled by shifting the charge of the Wilson line $\tilde{q}_s$ or q_{s-1} by $\pm 1/2$.

Under the interpretation of $\beta_{(a)}$ as position coordinates, the assumption (4.3.9) means that the 3D gauge node is at $\beta = 0$. Note that this assumption is not mandatory. One may start with a quiver realization in which the 3D gauge node is not at $\beta = 0$ and find necessary modifications, though the answer will not be as simple as the one given above.

The vortex loops in 3D $U(N)$ gauge theory with an anti-fundamental chiral multiplet can be studied in the same way. Depending on the boundary condition, the function $V_\beta(\widehat{\sigma})$ is given by (4.3.4) with

$$\begin{aligned} V_\beta^{\rm c\mathbf{1}}(\widehat{\sigma}) &= \prod_{\beta_i>0} 2\sinh\pi\Big(\widehat{\sigma}_i - \widehat{m} - \frac{irbQ}{2}\Big), \\ V_\beta^{\rm c\mathbf{2}}(\widehat{\sigma}) &= \prod_{\beta_i<0} \Big(2\sinh\pi\Big(\widehat{\sigma}_i - \widehat{m} + \frac{i(2-r)bQ}{2}\Big)\Big)^{-1}. \end{aligned} \tag{4.3.10}$$

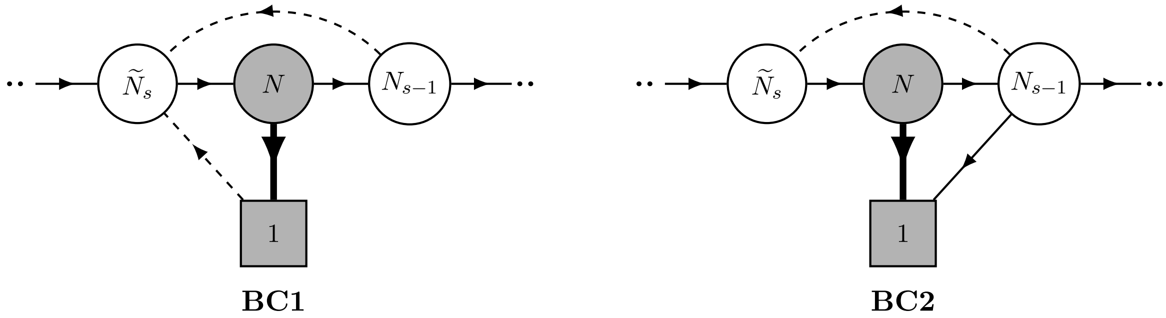

Figure 4.8 Quivers for the worldline theory of a vortex loop in 3D $U(N)$ gauge theory with a fundamental chiral multiplet satisfying **BC1** or **BC2**. In both diagrams, the 3D gauge and global symmetries $U(N) \times U(1)_{\mathrm{m}}$ are represented by shaded nodes and the 3D fundamental chiral multiplet is represented by a thick link.

The corresponding vortex worldline theories are given by the two quivers of Fig. 4.9. They are modifications of the quiver theory of Fig. 4.3 by adding

(**BC1**) a Fermi multiplet in the bifundamental of $U(N_{s-1}) \times U(1)_{\mathrm{m}}$ with $\mathrm{G} = -r$,

(**BC2**) a chiral multiplet in the bifundamental of $U(1)_{\mathrm{m}} \times U(\tilde{N}_s)$ with $\mathrm{G} = r - 2$.

Also, the charge of the Wilson line needs to be modified to take care of global anomaly.

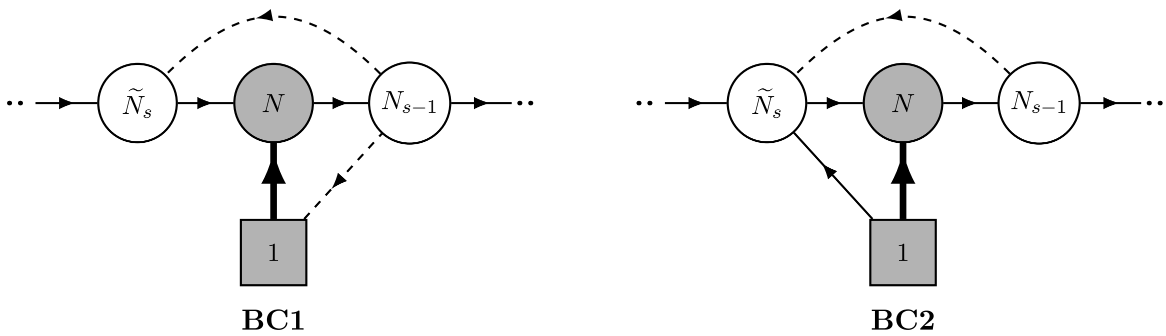

Figure 4.9 Addition of an anti-fundamental chiral multiplet to 3D $U(N)$ theory and the corresponding modification of the GLSM on vortex worldline.

Let us explain how we determined the orientation of the arrows for the 1D matter multiplets just added. For the cases with **BC2**, the added 1D chiral multiplets contribute to the denominator of Δ (4.2.6) and give rise to more poles. But those new poles must not contribute to the index. This determines the orientation of the arrow for the added chiral multiplets. For the case with **BC1**, the orientation of the arrow for Fermi multiplets has been determined from the consistency with SUSY enhancement. As we will discuss in the next section, when 3D bulk theory has $\mathcal{N} = 4$, the vortex worldline theory also has an enhanced $\mathcal{N} = 4$ SUSY.

Large mass limit. Integration of massive chiral multiplets in 3D sometimes yields an effective CS coupling [52, 53]. In the presence of vortex loop, it also gives rise to an effective Wilson line for the worldline theory. Let us study this effect in a simple example.

Consider a 3D $U(N)_k$ CS theory with one fundamental and one anti-fundamental chiral multiplets with the masses $m_\mathrm{f}, m_\mathrm{a}$ and R-charges $r_\mathrm{f}, r_\mathrm{a}$. They contribute the following one-loop determinant to the ellipsoid partition function (2.4.1):

$$\Delta^{\mathrm{c}}_{\text{1-loop}} = \prod_{i=1}^{N} s_b\Big(\frac{i(1-r_\mathrm{f})Q}{2} - \hat{\sigma}_i + \hat{m}_\mathrm{f}\Big) s_b\Big(\frac{i(1-r_\mathrm{a})Q}{2} + \hat{\sigma}_i - \hat{m}_\mathrm{a}\Big) . \tag{4.3.11}$$

By using the asymptotics of the double sine function

$$s_b(x) \sim \exp\frac{\pm i\pi}{2}\Big(x^2 + \frac{b^2+b^{-2}}{12}\Big) \qquad (\mathrm{Re}(x) \to \pm\infty)$$

and comparing with (2.3.4), one finds that the integration of the heavy chiral multiplets in the limit $m_\mathrm{f} \to \pm\infty$, $m_\mathrm{a} \to \mp\infty$ shifts the CS and FI couplings by

$$\delta k = \pm 1, \qquad \delta\zeta = \pm\frac{\widehat{m}_\mathrm{f} + \widehat{m}_\mathrm{a}}{2} \pm \frac{i(r_\mathrm{f} - r_\mathrm{a})Q}{4} . \tag{4.3.12}$$

Let us introduce a vortex loop with vorticity β and put the boundary condition **BC1** for the fundamental, **BC2** for the anti-fundamental chirals. As explained above, the 1D theory has an additional pair of chiral and Fermi multiplets in the anti-fundamental of $U(\tilde{N}_s)$. The added matters do not produce anomaly, so the $U(1)^{p-1}$ charge of the Wilson line may be chosen as (4.2.13). The one-loop determinant Δ of the worldline theory is modified by the factor

$$\prod_{\beta_i<0} \frac{2\sinh\pi(\widehat{\sigma}_i - \widehat{m}_\mathrm{f} + \frac{ir_\mathrm{f}bQ}{2})}{2\sinh\pi(\widehat{\sigma}_i - \widehat{m}_\mathrm{a} + \frac{i(2-r_\mathrm{a})bQ}{2})} \quad\longrightarrow\quad \exp\Big(\mp 2\pi\sum_{\beta_i<0}\widehat{\sigma}_i\Big). \tag{4.3.13}$$

This corresponds to the shift of the charge of the Wilson line $\tilde{q}_s$ by ∓ 1.

Here we recall that the charges (4.2.13) of the Wilson line was determined from the consistency with the relation $\lambda_i = k\beta_i$ in pure CS theory. However, after the massive matters are introduced and integrated out, the parameters $k, q_a, \tilde{q}_a$ will get corrected and (4.2.13) will no longer be satisfied. Taking account of this effect, perhaps one should regard λ_i or $(q_a, \tilde{q}_a)$ as more important label than β since they determine the value of BPS vortex loop observables more directly. But β still has an important role to set the pattern of gauge symmetry breaking and the orderings of unbroken gauge group factors.

Chapter 5

$\mathcal{N} = 4$ theories

In this chapter we extend our description of vortex loops to those in 3D $\mathcal{N} = 4$ theories. We will first find out the condition on the singular behavior of fields near 1/2 BPS vortex loops, and then identify the corresponding worldline quantum mechanics with 1D $\mathcal{N} = 4$ supersymmetry.

We begin by reviewing the basic properties of 3D $\mathcal{N} = 4$ gauge theories. For the theories on flat $\mathbb{R}^3$, the four sets of supercharges transform as a bispinor under the R-symmetry $SU(2)_{\mathrm{C}} \times SU(2)_{\mathrm{H}}$. We denote its Cartan generators as $\mathrm{J}^3_{\mathrm{C}}$ and $\mathrm{J}^3_{\mathrm{H}}$. A 3D $\mathcal{N} = 4$ vectormultiplet is made from an $\mathcal{N} = 2$ vectormultiplet $(A_m, \sigma, \lambda, \bar{\lambda}, D)$ and an adjoint chiral multiplet (ϕ, ψ, F). The three scalars $(\sigma, \phi, \bar{\phi})$, three auxiliary scalars $(D, F, \bar{F})$ and four spinors $(\lambda, \bar{\lambda}, \psi, \bar{\psi})$ form the representations $(\mathbf{3}, \mathbf{1})$, $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{2})$ of $SU(2)_{\mathrm{C}} \times SU(2)_{\mathrm{H}}$, respectively. In our convention ϕ has $\mathrm{J}^3_{\mathrm{C}} = 1$ whereas F has $\mathrm{J}^3_{\mathrm{H}} = 1$. The charges of the fields are summarized in Table 5.1.

field	A_m	σ	ϕ	$\bar{\phi}$	λ	$\bar{\lambda}$	ψ	$\bar{\psi}$	D	F	$\bar{F}$
$\mathrm{J}^3_{\mathrm{C}}$	0	0	$+1$	-1	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	0
$\mathrm{J}^3_{\mathrm{H}}$	0	0	0	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	$+1$	-1

Table 5.1 R-charges of $\mathcal{N} = 4$ vectormultiplet fields.

Let us turn to the theory on S^3. The SYM Lagrangian for an $\mathcal{N} = 4$ vectormultiplet is given by the sum of $\mathcal{L}_{\mathrm{YM}}$ for the vectormultiplet and $g^{-2}\mathcal{L}_{\mathrm{mat}}$ for the adjoint chiral multiplet in (1.3.14),(1.3.15). It is not $SU(2)_{\mathrm{C}} \times SU(2)_{\mathrm{H}}$ R-symmetric due to the coupling with the background auxiliary field. But when $\ell = \tilde{\ell} = f$ and the adjoint chiral multiplet has $r = 1$ the Lagrangian has a $\mathbb{Z}_2$ invariance:

$$\begin{aligned} &\mathcal{L}(A_m\,;\, \sigma, \phi, \bar{\phi}\,;\, \lambda, \bar{\lambda}, \psi, \bar{\psi}\,;\, D, F, \bar{F}\,;\, H) \\ &= \mathcal{L}(A_m\,;\, -\sigma, \bar{\phi}, \phi\,;\, \bar{\psi}, -\psi, -\bar{\lambda}, \lambda\,;\, D, -F, -\bar{F}\,;\, -H). \end{aligned} \tag{5.0.1}$$

This implies that $\mathcal{L}$ with $r = 1$ on a round S^3 has an enhanced supersymmetry: in addition to the original $\mathcal{N} = 2$ SUSY corresponding to the four independent solutions of (1.2.28), it has the second set of $\mathcal{N} = 2$ SUSY corresponding to four independent solutions of the same equation

(1.2.28) with H sign-flipped. The $U(1)$ R-charge of the original $\mathcal{N}=2$ SUSY is identified with $\mathrm{J}^3_\mathrm{C} - \mathrm{J}^3_\mathrm{H}$ because the fields ϕ, ψ, F have the charges $\mathrm{R}_{U(1)} = 1, 0, -1$. Similarly, the $U(1)$ R-charge of the second $\mathcal{N}=2$ SUSY is identified as $\mathrm{R}'_{U(1)} = -\mathrm{J}^3_\mathrm{C} - \mathrm{J}^3_\mathrm{H}$.

A hypermultiplet in a representation Λ of the gauge group consists of $\mathcal{N}=2$ chiral multiplets in the representations Λ and $\bar{\Lambda}$. We will denote the chiral scalars as $q, \tilde{q}$, and their spinor superpartners as $\chi, \tilde{\chi}$. It is known that $(q, \bar{\tilde{q}})$ form a doublet of $SU(2)_\mathrm{H}$ and $(\chi, \bar{\tilde{\chi}})$ form a doublet of $SU(2)_\mathrm{C}$. On S^3, these two chiral multiplets both need to have $r = 1/2$. Then the $\mathbb{Z}_2$ symmetry (5.0.1) of the theory on S^3 can be easily extended to hypermultiplet sector by identifying it with an element of $SU(2)_\mathrm{C} \times SU(2)_\mathrm{H}$. The charges of the hypermultiplet fields are summarized in Table 5.2.

field	q	$\bar{q}$	$\tilde{q}$	$\bar{\tilde{q}}$	χ	$\bar{\chi}$	$\tilde{\chi}$	$\bar{\tilde{\chi}}$
$\mathrm{R}_{U(1)}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$\mathrm{R}'_{U(1)}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$
J^3_C	0	0	0	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
J^3_H	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	0	0	0

Table 5.2 R-charges of hypermultiplet fields.

5.1 BPS boundary condition

Let us now turn to the definition of vortex loops. Consider first a vortex line stretching along the x^3-axis of flat $\mathbb{R}^3$. As in the cases with $\mathcal{N}=2$ SUSY, the gauge field behaves as

$$A \sim \beta \mathrm{d}\varphi, \qquad F_{12} = 2\pi\beta\delta^2(x^1, x^2) + \cdots .$$

The vortex configuration can be made half-BPS by turning on the $SU(2)_\mathrm{H}$-triplet auxiliary scalars $D^a = (F, \bar{F}, D)$ appropriately. The unbroken SUSY then corresponds to solutions of the BPS equation of the form

$$0 \;=\; \boldsymbol{Q}\lambda_{A\bar{B}} \;=\; F_{12}\gamma^{12}\xi_{A\bar{B}} - D^a\xi_{A\bar{C}}(\sigma^a)^{\bar{C}}{}_{\bar{B}}, \tag{5.1.1}$$

where $A, B, \cdots$ and $\bar{A}, \bar{B}, \cdots$ are doublet indices for $SU(2)_\mathrm{C}$ and $SU(2)_\mathrm{H}$ respectively, and σ^a is Pauli's matrix. It has nontrivial solutions if one sets, for example,

$$D^3 = D = iF_{12}\,.$$

The Lorentz symmetry $SU(2)_\mathrm{Lorentz}$ and the R-symmetry $SU(2)_\mathrm{C} \times SU(2)_\mathrm{H}$ are then broken to $U(1)_\mathrm{M} \times SU(2)_\mathrm{C} \times U(1)_{\mathrm{J}^3_\mathrm{H}}$, where M generates the rotation about the x^3-axis. Four of the eight supercharges corresponding to the SUSY parameter $\xi_{A\bar{B}}$ with $\gamma^3 = \pm 1$ ($\mathrm{M} = \pm\frac{1}{2}$) and $\mathrm{J}^3_\mathrm{H} = \mp\frac{1}{2}$ remain unbroken.

Let us next consider the theory on a round S^3 with a half-BPS vortex loop along $S^1_{(\tau)}$ at $\theta = 0$. Four of the eight supercharges are broken as in flat space. Two of the four unbroken supercharges correspond to the Killing spinors $\xi, \bar{\xi}$ of (1.2.32), and the other two correspond to new Killing spinors

$$\xi' = e^{\frac{i}{2}(\varphi-\tau)} \begin{pmatrix} \cos\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} \end{pmatrix}, \qquad \bar{\xi}' = e^{-\frac{i}{2}(\varphi-\tau)} \begin{pmatrix} -i\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{5.1.2}$$

These four Killing spinors satisfy

$$\nabla_m \xi = \frac{i}{2\ell}\gamma_m \xi, \quad \nabla_m \bar{\xi} = \frac{i}{2\ell}\gamma_m \bar{\xi}, \quad \nabla_m \xi' = -\frac{i}{2\ell}\gamma_m \xi', \quad \nabla_m \bar{\xi}' = -\frac{i}{2\ell}\gamma_m \bar{\xi}'.$$

One can check that the new Killing spinors $\xi', \bar{\xi}'$ have $\mathrm{M} = \pm\frac{1}{2}$, so the flat space analysis implies they have $\mathrm{J}^3_{\mathrm{H}} = \mp\frac{1}{2}$. The quantum numbers of the four Killing spinors are thus determined as in Table 5.3. The $\mathbb{Z}_2$ transformation (5.0.1) acts as

$$\xi \leftrightarrow \xi', \qquad \bar{\xi} \leftrightarrow \bar{\xi}'. \tag{5.1.3}$$

Killing spinor	$\mathrm{R}_{U(1)}$	$\mathrm{R}'_{U(1)}$	$\mathrm{J}^3_{\mathrm{C}}$	$\mathrm{J}^3_{\mathrm{H}}$	$-i\mathcal{L}_{\partial_\tau}$	$-i\mathcal{L}_{\partial_\varphi}$
ξ	$+1$	0	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$
$\bar{\xi}$	-1	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
ξ'	0	$+1$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$
$\bar{\xi}'$	0	-1	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$

Table 5.3 Quantum numbers of Killing spinors on S^3.

Boundary condition on fluctuations. It remains to check if there is a set of boundary conditions on the fluctuation of fields preserving 1/2 of the $\mathcal{N} = 4$ SUSY. We continue to work with a vortex loop in S^3 winding along the $S^1_{(\tau)}$ at $\theta = 0$.

Let us first study the fluctuation of $\mathcal{N} = 4$ vectormultiplet using the decomposition into $\mathcal{N} = 2$ multiplets. According to what we found in Section 2.1 for the fluctuation of $\mathcal{N} = 2$ vectormultiplet fields, $\xi\lambda, \bar{\xi}\bar{\lambda}$ may diverge mildly but $\xi\bar{\lambda}, \bar{\xi}\lambda$ must be finite near the vortex loop. The boundary condition also preserves the SUSY corresponding to $\xi', \bar{\xi}'$ if it respects the $\mathbb{Z}_2$ invariance (5.0.1) and (5.1.3). So $\xi'\bar{\psi}, \bar{\xi}'\psi$ may diverge but $\xi'\psi, \bar{\xi}'\bar{\psi}$ must be finite. Here one can replace ξ' by ξ (and similarly $\bar{\xi}'$ by $\bar{\xi}$) because they are proportional to each other along the vortex loop. The resulting boundary conditions on ψ and $\bar{\psi}$ imply that the $\mathcal{N} = 2$ adjoint chiral multiplet must obey **BC1**.

To be fully explicit, let us list the boundary condition for all the fields in an $\mathcal{N} = 4$ vectormultiplet near a BPS vortex loop:

$$\begin{array}{llll} \xi\gamma^m\xi\delta A_m,\ \bar{\xi}\gamma^m\bar{\xi}\delta A_m, & \xi\lambda,\ \xi\bar{\psi},\ \bar{\xi}\bar{\lambda},\ \bar{\xi}\psi, & F,\ \bar{F} & \text{may diverge,} \\ \bar{\xi}\gamma^m\xi\delta A_m,\ \delta\sigma,\ \phi,\ \bar{\phi}, & \xi\bar{\lambda},\ \xi\psi,\ \bar{\xi}\lambda,\ \bar{\xi}\bar{\psi}, & \delta D & \text{must be finite.} \end{array} \tag{5.1.4}$$

This preserves the SUSY corresponding to $\xi, \bar{\xi}$ as well as $\xi', \bar{\xi}'$.

The above form of boundary condition can also be used for a vortex line lying along, say, the x^3-axis of flat $\mathbb{R}^3$. In that case ξ, ξ' are eigenspinors of $\gamma^3 = 1$ and $\bar{\xi}, \bar{\xi}'$ are eigenspinors of $\gamma^3 = -1$. The above set of boundary conditions is clearly consistent with the unbroken $SU(2)_{\mathrm{C}}$ symmetry.

A hypermultiplet in a representation Λ consists of an $\mathcal{N} = 2$ chiral multiplet q, χ in Λ and another chiral multiplet $\tilde{q}, \tilde{\chi}$ in $\bar{\Lambda}$. To preserve the SUSY corresponding to ξ and $\bar{\xi}$, each of the two chiral multiplets must obey the boundary condition **BC1** or **BC2**. Then, as in the previous paragraph, one can argue that the unbroken SUSY enhances if the boundary condition respects the $SU(2)_{\mathrm{C}}$ symmetry. Recall that, whichever boundary conditions we choose, the fields in the representation Λ are divided into four groups of cohomological variables as follows:

$$\begin{aligned} q, \xi\chi \ \in \ \mathcal{H} \ \ &\underset{\bar{\mathcal{J}}}{\overset{\mathcal{J}}{\rightleftarrows}} \ \ \mathcal{H}' \ \ni \ \bar{\xi}\chi, \\ \bar{\tilde{q}}, \xi\bar{\tilde{\chi}} \ \in \ \tilde{\mathcal{H}}^* \ \ &\underset{(\bar{\tilde{\mathcal{J}}})^\dagger}{\overset{(\tilde{\mathcal{J}})^\dagger}{\rightleftarrows}} \ \ \tilde{\mathcal{H}}'^* \ \ni \ \xi\bar{\tilde{\chi}}. \end{aligned} \tag{5.1.5}$$

Here the differential operators $\tilde{\mathcal{J}}, \bar{\tilde{\mathcal{J}}}$ are defined in the same way as $\mathcal{J} \equiv i\bar{\xi}\gamma^m\bar{\xi}\nabla_m$ and $\bar{\mathcal{J}} \equiv -i\xi\gamma^m\xi\nabla_m$ using the covariant derivative for fields in $\bar{\Lambda}$. So in fact $(\tilde{\mathcal{J}})^\dagger = \bar{\mathcal{J}}$ and $(\bar{\tilde{\mathcal{J}}})^\dagger = \mathcal{J}$. Hence one can preserve $SU(2)_{\mathrm{C}}$ by imposing the same boundary condition on $\xi\chi$ and $\xi\bar{\tilde{\chi}}$, and similarly on $\bar{\xi}\chi$ and $\bar{\xi}\bar{\tilde{\chi}}$, which form doublets. This leads us to conclude that there are the following two BPS boundary conditions on a hypermultiplet:

- $\xi\chi, \xi\bar{\tilde{\chi}}$ are finite but $\bar{\xi}\chi, \bar{\xi}\bar{\tilde{\chi}}$ may diverge near the vortex loop. Namely, the chiral multiplet (q, χ) obeys **BC1** and $(\tilde{q}, \tilde{\chi})$ obeys **BC2**.
- The opposite of the above. Namely, (q, χ) obeys **BC2** and $(\tilde{q}, \tilde{\chi})$ obeys **BC1**.

Note that our result is similar to the one obtained in [54]. There the fluctuation of fields with more general (i.e. not necessarily mild) singular behavior near vortex lines is considered.

5.2 $\mathcal{N} = 4$ SUSY quantum mechanics

Let us next turn to the study of the vortex worldline theories. For a straight vortex line in a flat $\mathbb{R}^3$, the worldline theory has a global symmetry $SU(2)_{\mathrm{C}} \times U(1)_{\mathrm{J}^3_{\mathrm{H}}} \times U(1)_{\mathrm{M}}$. The four unbroken supercharges transform under its $SU(2) \times U(1)$ subgroup as two $SU(2)$-doublets of $U(1)$ charge ± 1. The 1D $\mathcal{N} = 4$ SUSY with this R-symmetry is a dimensional reduction of the 4D $\mathcal{N} = 1$ SUSY.

A 1D $\mathcal{N} = 4$ vectormultiplet is made from an $\mathcal{N} = 2$ vectormultiplet $(A_t, \sigma, \lambda, \bar{\lambda}, D)$ and an adjoint chiral multiplet (ϕ, ψ). The quantum numbers of the fields are determined as in Table 5.4 from the fact that $\epsilon, \bar{\epsilon}$ in the transformation rules (3.2.14) and (4.1.1) carry the same quantum numbers as $\xi, \bar{\xi}$. The $U(1)$ R-charge of 1D $\mathcal{N} = 4$ SUSY is identified with a linear combination

$$c_1 \mathrm{J}^3_{\mathrm{H}} + c_2 \mathrm{M}. \quad (c_2 - c_1 = 2)$$

field	A_t	σ	ϕ	$\bar{\phi}$	λ	$\bar{\lambda}$	ψ	$\bar{\psi}$	D	ϵ	$\bar{\epsilon}$
$\mathrm{J}^3_{\mathrm{C}}$	0	0	$+1$	-1	$+\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$-\frac{1}{2}$
$\mathrm{J}^3_{\mathrm{H}}$	0	0	0	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$+\frac{1}{2}$
M	0	0	0	0	$+\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$+\frac{1}{2}$	0	$+\frac{1}{2}$	$-\frac{1}{2}$

Table 5.4 Quantum numbers of $\mathcal{N} = 4$ vectormultiplet fields on the vortex worldline.

For the computation of Witten index, one chooses a pair of supercharges (such as the pair we have been using in the previous chapter) that generate an $\mathcal{N} = 2$ subalgebra. The index can be generalized by twisting the periodic boundary condition of fields by global symmetries that commute with the chosen supercharges. Of particular importance is the symmetry generated by $\mathrm{G} \equiv \mathrm{J}^3_{\mathrm{C}} - \mathrm{J}^3_{\mathrm{H}} - 2\mathrm{M}$, as it shows up in the Witten index for vortex loops inside S^3. This G was already introduced in the previous chapter at (4.3.2) as a non-R global symmetry of $\mathcal{N} = 2$ SUSY theories. One can easily find from Table 5.4 that $\mathcal{N} = 2$ vectormultiplet has $\mathrm{G} = 0$ while the adjoint chiral multiplet has $\mathrm{G} = +1$.

A 1D $\mathcal{N} = 4$ chiral multiplet is made from an $\mathcal{N} = 2$ chiral multiplet (q, χ) and a Fermi multiplet (η, F) in the same representation of the gauge group. The quantum numbers of fields under $\mathrm{J}^3_{\mathrm{H}}$ and M are constrained only by the requirement that the fermions (χ, η) form an $SU(2)$-doublet, so generally they take values as summarized in Table 5.5. This implies that, if q and χ have $\mathrm{G} = g$, then η and F should have $\mathrm{G} = g + 1$. We call such a set of fields an $\mathcal{N} = 4$ chiral multiplet of $\mathrm{G} = g$.

field	q	χ	η	F
$\mathrm{J}^3_{\mathrm{C}}$	0	$-\frac{1}{2}$	$+\frac{1}{2}$	0
$\mathrm{J}^3_{\mathrm{H}}$	a	$a + \frac{1}{2}$	$a + \frac{1}{2}$	$a + 1$
M	b	$b - \frac{1}{2}$	$b - \frac{1}{2}$	$b - 1$

Table 5.5 Quantum numbers of $\mathcal{N} = 4$ chiral multiplet fields on the vortex worldline.

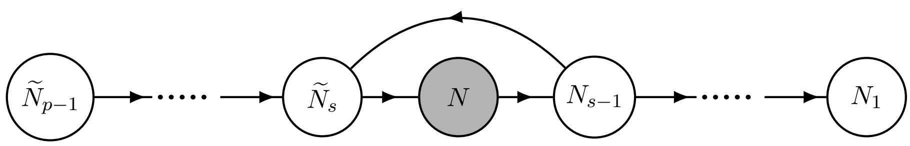

Figure 5.1 1D $\mathcal{N} = 4$ quiver diagram describing the GLSM for a vortex loop in 3D $\mathcal{N} = 4$ $U(N)$ pure SYM. The white nodes and solid lines represent 1D $\mathcal{N} = 4$ vector and chiral multiplets.

Examples. As the most basic example of vortex loops in 3D $\mathcal{N}=4$ theories, let us consider those in $\mathcal{N}=4$ $U(N)$ pure SYM. The worldline theory is a special case ($m=0, r=1$ and **BC1**) of the quiver GLSM studied in Section 4.3.2. In 1D $\mathcal{N}=2$ terminology, it is a gauge theory with the 1D and 3D gauge groups

$$U(\tilde{N}_{p-1})\times\cdots U(\tilde{N}_s)\times U(N)_{(3\mathrm{D})}\times U(N_{s-1})\times\cdots U(N_1).$$

The $\mathcal{N}=2$ vectormultiplet for each 1D gauge group factor is paired with an adjoint chiral multiplet with $\mathrm{G}=1$ to form an $\mathcal{N}=4$ vectormultiplet. For each pair of neighboring gauge group factors one has a pair of bifundamental $\mathcal{N}=2$ chiral and Fermi multiplets of $\mathrm{G}=0$ and 1, which form an $\mathcal{N}=4$ chiral multiplet of $\mathrm{G}=0$. The theory also has a pair of a chiral and Fermi multiplets of $G=-1,0$ in the bifundamental of $U(N_{s-1})\times U(\tilde{N}_s)$, which form an $\mathcal{N}=4$ chiral multiplet with $\mathrm{G}=-1$. The field content is described by the $\mathcal{N}=4$ quiver diagram of Fig. 5.1. The FI couplings for $U(N_a)$ are all negative while those for $U(\tilde{N}_a)$ are all positive. The theory has no Wilson line since it is free of global anomaly and one cannot turn on 3D CS coupling without breaking SUSY to $\mathcal{N}\leq 3$.

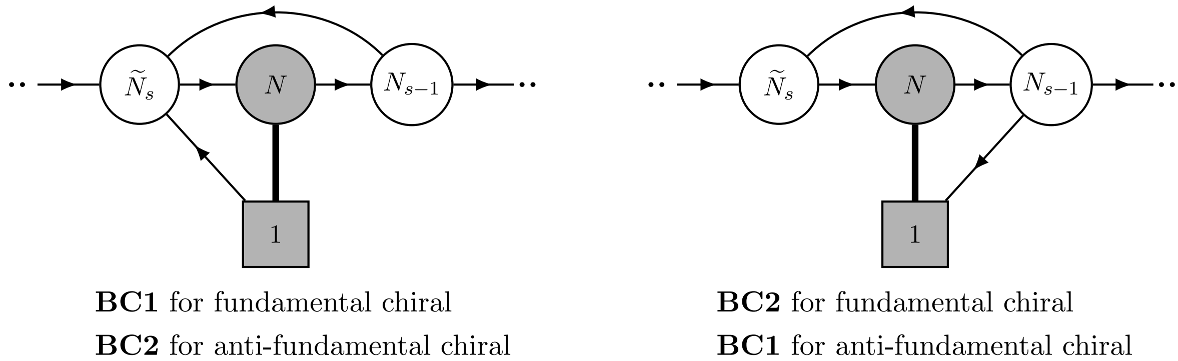

Figure 5.2 Addition of a fundamental hypermultiplet to 3D $\mathcal{N}=4$ $U(N)$ theory and the corresponding modification of the vortex worldline GLSM.

The next simplest are the vortex loops in 3D $\mathcal{N}=4$ $U(N)$ gauge theory with a fundamental hypermultiplet. The worldline theory is obtained by adding some more fields to the theory described previously according to the discussion of Section 4.3.3. The corresponding quiver diagram is presented in Fig. 5.2. As was explained in Section 5.1, there are two consistent boundary conditions on the hypermultiplet, which result in two different modification of the quiver diagram of Fig. 5.1. The added $\mathcal{N}=4$ chiral multiplet is either in the anti-fundamental of $U(\tilde{N}_s)$ or in the fundamental of $U(N_{s-1})$, and it has $\mathrm{G}=-3/2$ in both cases. Note that the model agrees with the one discussed in [11, 20] if the 3D $U(N)$ gauge node is at either end of the linear quiver.

Background fields for vortex loops in S^3. Let us explain what kind of background fields appear on the worldline of vortex loops in 3D $\mathcal{N}=4$ gauge theories on S^3.

We recall that the 1D $\mathcal{N}=2$ SUSY of the vortex worldline theory was defined in accordance with the 3D SUSY acting on cohomological variables. For vortex loops of $\mathcal{N}=4$ theory on S^3, the square of the supercharge is given by (here t is the worldline coordinate of period 2π)

$$\boldsymbol{Q}^2_{(1\mathrm{D})} \sim -\partial_t + \sigma + iA_t + i(\mathrm{J}^3_\mathrm{C} - \mathrm{J}^3_\mathrm{H} - 2\mathrm{M}) + (\text{3D vectormultiplet fields}). \tag{5.2.1}$$

$\mathcal{N}=4$ theories on S^3 have the second set of $\mathcal{N}=2$ SUSY corresponding to the Killing spinors $\xi', \bar{\xi}'$ (5.1.2). It can be used to define the second 1D supercharge $\boldsymbol{Q}'_{(1\mathrm{D})}$ which squares to

$$\boldsymbol{Q}'^2_{(1\mathrm{D})} \sim -\partial_t - \sigma + iA_t - i(-\mathrm{J}^3_\mathrm{C} - \mathrm{J}^3_\mathrm{H} - 2\mathrm{M}) + (\text{3D vectormultiplet fields})', \tag{5.2.2}$$

where the prime on the 3D vectormultiplet fields stands for the $\mathbb{Z}_2$ action defined in (5.0.1). Here one needs to be careful for the fact that the two supercharges are defined by identifying different sets of cohomological variables as 1D multiplets. The set of 1D variables on which $\boldsymbol{Q}_{(1\mathrm{D})}$ acts as (3.2.14) or (4.1.1) is therefore different from the set on which $\boldsymbol{Q}'_{(1\mathrm{D})}$ acts the same way. But the two sets of variables are related by a simple "gauge transformation" as we now explain.

Let Φ be a cohomological variable made of 3D fields and $\xi, \bar{\xi}$ such as Ψ that we considered in (4.3.1), and Φ' the same cohomological variable with $(\xi, \bar{\xi})$ replaced by $(\xi', \bar{\xi}')$. Using the quantum number of Killing spinors listed in Table 5.3 and the fact that cohomological variables are all 3D Lorentz scalar, one generally finds

$$\partial_\tau \Phi = i(\mathrm{H}-\mathrm{M})\Phi, \qquad \partial_\tau \Phi' = i(\mathrm{H}+\mathrm{M})\Phi'. \tag{5.2.3}$$

So the two cohomological variables are related by

$$\Phi' = e^{2i\mathrm{M}\tau}\Phi. \tag{5.2.4}$$

The gauge transformation relating the two sets of 1D variables explained above is given by the same formula. Therefore, when considering the action of $\boldsymbol{Q}'^2_{(1\mathrm{D})}$ on Φ instead of Φ', the RHS of (5.2.2) has to be shifted by $-2i\mathrm{M}$ due to the above gauge transformation. The value of the background 1D vectormultiplet field is thus determined as follows.

$$\begin{array}{rcl} \sigma^{\mathrm{bg}} + iA^{\mathrm{bg}}_t &=& i(\mathrm{J}^3_\mathrm{C} - \mathrm{J}^3_\mathrm{H} - 2\mathrm{M}), \\ -\sigma^{\mathrm{bg}} + iA^{\mathrm{bg}}_t &=& -i(-\mathrm{J}^3_\mathrm{C} - \mathrm{J}^3_\mathrm{H}), \end{array} \quad \therefore \quad \begin{array}{rcl} A^{\mathrm{bg}}_t &=& \mathrm{J}^3_\mathrm{C} - \mathrm{M}, \\ \sigma^{\mathrm{bg}} &=& -i\mathrm{J}^3_\mathrm{H} - i\mathrm{M}. \end{array}$$

Thus we recovered the result in Section 5.2 of [11] using a slightly different argument.

Chapter 6

Conclusion

In this thesis we studied different descriptions of BPS vortex loops in 3D $\mathcal{N} = 2$ gauge theories and derived exact formulae for their expectation values on an ellipsoid.

In Chapter 1 and Chapter 2 we reviewed $\mathcal{N} = 2$ SUSY gauge theories and the exact formula for the partition function on 3D ellipsoid using localization techniques. The expectation value of the vortex loop with the definition based on the singular gauge field were also computed there. However, as discussed in Chapter 3, we realized that the result needed to be modified in order to respect the well-known correspondence between Wilson and vortex loops [16]. We found that our argument has to be modified with regard to the following two points.

(i) One was that we missed another boundary S_{QM}, which is necessary so that the variation principle lead to the equation of motion and the desired boundary condition. By employing the coadjoint orbit quantization for the definition of Wilson loops, we demonstrated the equivalence between Wilson and vortex loops, particularly in bosonic Pure CS theories.

(ii) The another point was that in $\mathcal{N} = 2$ theories the rule of correspondence contains the unwanted parameter shift $\lambda \to \lambda + \tilde{\rho}$. We resolved this shift by relating it to the global anomaly of 1D theory on vortex worldline and canceling it by Wilson lines.

On the other hand, our analysis of the boundary term revealed that vortex loops can also be defined as a quantum mechanics on a loop interacting with the field theory in 3D space.

To extend the correspondence of the two definitions of vortex loops to a wider class of $\mathcal{N} = 2$ theories, we developed the descriptions of coadjoint orbit quantum mechanics as quiver GLSMs. The index of the GLSM was computed by JK-residue prescription and we explicitly confirmed the correspondence between the two descriptions. We also identified the extensions of these GLSMs that incorporate the addition of various matter chiral multiplets on the vortex background. This was done for the matters in the adjoint, fundamental and anti-fundamental representations of $U(N)$.

As another extension, we studied vortex loops in $\mathcal{N} = 4$ theories consisting of vector and hypermultiplets. By analyzing them using decomposition into $\mathcal{N} = 2$ multiplets we found that, in order for the vortex loop to preserve 1/2 SUSY, the adjoint chiral multiplet (which is a part of $\mathcal{N} = 4$ vectormultiplet) must always obey **BC1**. And we identified the the corresponding worldline quantum mechanics with 1D $\mathcal{N} = 4$ supersymmetry.

However, there still remain some unclear points. Regarding the correspondence between Wilson and vortex loops in terms of our description, (i) is much clear, whereas (ii) is not: what the Wilson line corresponds to in the description specified by singular behavior of the field is not clear. In Chapter 2 we performed exact path integration on singular vortex backgrounds, but the result did not respect the relation [16] claimed by Moore and Seiberg and a parameter shift that could be interpreted as an anomaly appeared. It might be interesting to understand the source and resolutions of this anomaly without moving to the description in terms of 1D-3D coupled systems.

As noted above, when analyzing $\mathcal{N} = 4$ theories we found the $\mathcal{N} = 2$ adjoint chiral multiplet must always obey **BC1**, though the SUSY on the vortex worldline seems to enhance for both choices of boundary conditions. It may be the case that the mechanism of SUSY enhancement is different for vortex loops in the ABJM model (for a recent work, see [55]) or other CS-matter theories with $\mathcal{N} \geq 4$ SUSY that were classified in [56, 57].

Finally, it should be noted that we were able to reproduce the worldline theory for only a part of the vortex loops that were identified in [11] as the mirror of Wilson loops. The main limitation for our analysis arises from that the function $V_\beta(\hat{\sigma})$ has to be expressed as a sum over elements of $\mathcal{W}/\mathcal{W}_K$ as in (4.2.2) or (4.3.4). This imposes a constraint on the set of poles contributing to the JK-residue integral for the index $I(\hat{\sigma})$. On the other hand, [11] has examples of vortex loops for which the index receives contributions from more poles. Perhaps this means there are more vortex loops defined by worldline quantum mechanics than those described by singular behavior of fields. Or it might be the case that we could reproduce more vortex loops in [11] by relaxing the assumption of small β (2.4.7). In either case, more thorough study of the correspondence is needed for a full understanding.

acknowledgment

I am grateful to my supervisor Prof. Kazuo Hosomichi for his constructive suggestions and patient guidance over the past five years. I would like to express gratitude to Prof. Taichiro Takagi and Dr. Seiji Sakoda for useful discussion and advice.

I give special thanks to my wife for her great support.

Appendix A

Killing spinors

A.1 $\mathbb{R}^3$

The simplest three-dimensional manifold is $\mathbb{R}^3$. The metric in cylindrical coordinate is

$$ds^2 = dr^2 + r^2 d\varphi^2 + dt^2, \qquad e^1 = dr, \quad e^2 = rd\varphi, \quad e^3 = dt. \tag{A.1.1}$$

This coordinate system is suited for studying vortex operators extending in t-direction at $r = 0$. By using (1.2.19), it can be easily shown that the only non-zero component of the spin connection is

$$\Omega^{12} = -d\varphi. \tag{A.1.2}$$

If the background fields V_m, H, K_m are zero, the Killing spinor equations (1.2.28) become

$$\begin{aligned} \nabla_m \xi &= 0, \\ \nabla_m \bar{\xi} &= 0, \end{aligned} \tag{A.1.3}$$

which mean that $\xi, \bar{\xi}$ are covariantly constant. In this case, by substituting (A.1.2) into the equations above, one finds the only nontrivial equations are the φ components:

$$\begin{aligned} \partial_\varphi \xi &= \frac{i}{2}\gamma^3 \xi, \\ \partial_\varphi \bar{\xi} &= \frac{i}{2}\gamma^3 \bar{\xi}. \end{aligned} \tag{A.1.4}$$

Each equation has two independent solutions, so (A.1.4) has four independent solutions. Two of them will be of particular importance, since they correspond to the SUSY preserved by certain 1/2-BPS line operators. We choose them to be eigenspinors of $\gamma^3 = +1, -1$. Their explicit form is

$$\xi = \begin{pmatrix} e^{\frac{i}{2}\varphi} \\ 0 \end{pmatrix}, \qquad \bar{\xi} = \begin{pmatrix} 0 \\ e^{-\frac{i}{2}\varphi} \end{pmatrix}, \tag{A.1.5}$$

Note that they are normalized so that $\bar{\xi}\xi = -1$. The Killing vector is then

$$v = \bar{\xi}\gamma^m \xi \partial_m = -\partial_t. \tag{A.1.6}$$

Note that the γ^3-eigenvalues of $\xi, \bar{\xi}$ are $+1, -1$, respectively, on the t-axis in which we intend to insert a vortex. The same selection is made for the other examples discussed below.

A.2 S^3 and ellipsoid

Some characteristic properties of ellipsoid are described here. An ellipsoid is defined by its embedding in $\mathbb{R}^4$ as follows [5]:

$$\frac{x_1^2+x_2^2}{\tilde{\ell}^2}+\frac{x_3^2+x_4^2}{\ell^2}=1, \qquad ds^2=dx_1^2+dx_2^2+dx_3^2+dx_4^2. \tag{A.2.1}$$

Moving from cartesian coordinate to polar coordinate by substituting (x_1, x_2, x_3, x_4) for $(\cos\theta\cos\varphi$, $\cos\theta\sin\varphi$, $\sin\theta\cos\tau$, $\sin\theta\sin\tau)$, a set of dreibein is expressed as follows:

$$e^1=f(\theta)d\theta, \quad e^2=\tilde{\ell}\sin\theta d\varphi, \quad e^3=\ell\cos\theta d\tau, \quad f(\theta)=\sqrt{\tilde{\ell}^2\sin^2\theta+\ell^2\cos^2\theta}. \tag{A.2.2}$$

On ellipsoid, non-zero components of the spin connection are

$$\Omega^{31}=-\frac{\ell}{f}\sin\theta d\tau, \qquad \Omega^{12}=-\frac{\tilde{\ell}}{f}\cos\theta d\varphi. \tag{A.2.3}$$

To solve the Killing spinor equation (1.2.28), we begin by writing each of its component separately as follows,

$$(\partial_\theta-iV_\theta)\,\xi=i\frac{f}{2}\gamma^1\kappa, \tag{A.2.4}$$

$$\left(\partial_\varphi-\frac{i}{2}\frac{\tilde{\ell}}{f}\cos\theta\gamma^3-iV_\varphi\right)\xi=\frac{i}{2}\tilde{\ell}\sin\theta\gamma^2\kappa, \tag{A.2.5}$$

$$\left(\partial_\tau-\frac{i}{2}\frac{\ell}{f}\sin\theta\gamma^2-iV_\tau\right)\xi=\frac{i}{2}\ell\cos\theta\gamma^3\kappa. \tag{A.2.6}$$

Here V_m is a suitable background U(1) gauge field. Let us, however, first solve the above equations with gauge field $V_m=0$ and $f=\tilde{\ell}=\ell$. This is just looking for Killing spinors on round S^3, but we will need them later when finding supersymmetric ellipsoid background. For this background one can solve (A.2.4) for κ:

$$\kappa=-2i\ell^{-1}\gamma^1\partial_\theta\xi. \tag{A.2.7}$$

Futhermore, we assume that ξ has definite φ- and τ-momenta.

$$\hat{\xi}_{st}=e^{\frac{i}{2}(s\varphi+t\tau)}\cdot\chi(\theta) \qquad (s,t=\pm1). \tag{A.2.8}$$

Then (A.2.5) and (A.2.6) can be written as

$$\chi=-2s\gamma^3\left(\frac{1}{2}\cos\theta-\sin\theta\partial_\theta\right)\chi \tag{A.2.9}$$

$$\chi=-2t\gamma^2\left(\frac{1}{2}\sin\theta+\cos\theta\partial_\theta\right)\chi. \tag{A.2.10}$$

Taking the difference of the two equations and then multiplying $(\gamma^3 s\sin\theta+\gamma^2 t\cos\theta)$ one obtains

$$\partial_\theta\chi=\frac{i}{2}st\gamma^1\chi, \tag{A.2.11}$$

which is solved by

$$\hat{\xi}_{st} = e^{\frac{i}{2}(s\varphi+t\tau)} \begin{pmatrix} Ae^{\frac{i}{2}\theta} + Be^{-\frac{i}{2}\theta} \\ st\left(Ae^{\frac{i}{2}\theta} - Be^{-\frac{i}{2}\theta}\right) \end{pmatrix}, \tag{A.2.12}$$

for arbitrary A, B. Substituting this into (A.2.9) or (A.2.10) one finds an additional condition $A = -stB$. Thus we obtain four independent solutions:

$$\begin{aligned} \hat{\xi}_{++}(A = B = 1/2) &= e^{\frac{i}{2}(\varphi+\tau)} \begin{pmatrix} \cos\frac{\theta}{2} \\ i\sin\frac{\theta}{2} \end{pmatrix}, \\ \hat{\xi}_{--}(A = -B = 1/2) &= e^{-\frac{i}{2}(\varphi+\tau)} \begin{pmatrix} i\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \end{aligned} \tag{A.2.13}$$

$$\begin{aligned} \hat{\xi}_{+-}(A = B = 1/2) &= e^{\frac{i}{2}(\varphi-\tau)} \begin{pmatrix} \cos\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} \end{pmatrix}, \\ \hat{\xi}_{-+}(A = -B = 1/2) &= e^{-\frac{i}{2}(\varphi-\tau)} \begin{pmatrix} -i\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \end{aligned} \tag{A.2.14}$$

where A, B are determined so that $\bar{\xi}\xi = \bar{\xi}'\xi' = -1$. And the Killing vector is

$$\bar{\xi}\gamma^a\xi = (0, -\sin\theta, -\cos\theta), \qquad v = -\frac{1}{\ell}\left(\partial_\varphi + \partial_\tau\right). \tag{A.2.15}$$

It is easy to confirm that $\hat{\xi}_{++}$, $\hat{\xi}_{--}$ are the solutions for $H = \frac{1}{\ell}$ with $V_m = K_m = 0$, whereas $\hat{\xi}_{+-}$, $\hat{\xi}_{-+}$ are the solutions for $H = -\frac{1}{\ell}$.

So far we have been looking at the Killing spinors for $f = \tilde{\ell} = \ell$, i.e. round S^3, but what we actually need is that for $\tilde{\ell} \neq \ell$, i.e. squashed S^3. For generic $\tilde{\ell} \neq \ell$ the Killing spinor equation has no solution unless background gauge field $V_m = 0$ is turned on. We determine V_m so that (A.2.13) or (A.2.14) remain solutions even after ellipsoidal deformation. Solving (A.2.4) for κ and using the result to eliminate κ in (A.2.5) and (A.2.6), one obtains

$$-2is\frac{f}{\tilde{\ell}}\left(i\frac{s}{2} - iV_\varphi\right)\chi = -2s\gamma^2\left(\frac{1}{2}\cos\theta - \sin\theta(\partial_\theta - V_\theta)\right)\chi \tag{A.2.16}$$

$$-2it\frac{f}{\ell}\left(i\frac{t}{2} - iV_\tau\right)\chi = -2t\gamma^1\left(\frac{1}{2}\sin\theta + \cos\theta(\partial_\theta - V_\theta)\right)\chi. \tag{A.2.17}$$

By comparing these with (A.2.9) and (A.2.10) one finds that, for the Killing spinor on the round S^3 (A.2.12) to remains solutions, the coefficients of χ on the left-hand sides should be one, and V_θ has to be zero. Therefore the background gauge field is

$$V^{st} = \frac{s}{2}\left(1 - \frac{\tilde{\ell}}{f}\right)d\varphi + \frac{t}{2}\left(1 - \frac{\ell}{f}\right)d\tau. \tag{A.2.18}$$

This means that if one choose V^{++} as the background gauge field, then $\xi = \hat{\xi}_{++}$ and $\bar{\xi} = \hat{\xi}_{--}$ define the rigid SUSY on the ellipsoid. Throughout this thesis we work with this choice for the supersymmetric ellipsoid background, namely we choose

$$V = \frac{1}{2}\left(1 - \frac{\tilde{\ell}}{f}\right)d\varphi + \frac{1}{2}\left(1 - \frac{\ell}{f}\right)d\tau, \quad H = \frac{1}{f}, \quad K_m = 0, \tag{A.2.19}$$

as the background fields. The Killing vector is

$$\bar{\xi}\gamma^a\xi = (0, -\sin\theta, -\cos\theta), \qquad v = -\frac{1}{\tilde{\ell}}\partial_\varphi - \frac{1}{\ell}\partial_\tau. \tag{A.2.20}$$

If we chose instead V^{+-} as the background gauge field, then the SUSY would be defined by $\xi = \hat{\xi}_{+-}$ and $\bar{\xi} = \hat{\xi}_{-+}$.

A.3 $S^2 \times S^1$

Another important background with rigid supersymmetry is $S^2 \times S^1$ which leads to the path integral definition of the 3D superconformal index [31, 58]. S^2 is parametrized by a spherical polar coordinate θ, φ $(0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi)$ as usual. So the metric is

$$ds^2 = \tilde{\ell}^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) + d\tau^2, \qquad e^1 = \tilde{\ell}d\theta, \quad e^2 = \tilde{\ell}\sin\theta d\varphi, \quad e^3 = d\tau, \tag{A.3.1}$$

where $\tau \sim \tau + 2\pi\ell$. Non-zero components of spin connections are

$$\Omega^{12} = -\cos\theta d\varphi, \quad \Omega^{23} = \Omega^{31} = 0. \tag{A.3.2}$$

In components, the Killing spinor equation is as follows:

$$(\partial_\theta - iV_\theta)\,\xi = \frac{i}{2}\tilde{\ell}\gamma^1\kappa, \tag{A.3.3}$$

$$\left(\partial_\varphi - \frac{i}{2}\cos\theta\gamma^3 - iV_\varphi\right)\xi = \frac{i}{2}\tilde{\ell}\sin\theta\gamma^2\kappa, \tag{A.3.4}$$

$$(\partial_\tau - iV_\tau)\,\xi = \frac{i}{2}\gamma^3\kappa. \tag{A.3.5}$$

Let us focus on the first two equations. If $V_\theta = V_\varphi = 0$, (A.3.3) and (A.3.4) imply

$$\left(\partial_\varphi - \frac{i}{2}\cos\theta\gamma^3\right)\xi = -i\sin\theta\gamma^3\partial_\theta\xi. \tag{A.3.6}$$

Taking the ansatz

$$\hat{\xi}_s \equiv e^{\frac{i}{2}s\varphi}\cdot\chi(\theta)\cdot\rho(\tau), \qquad (s = \pm 1) \tag{A.3.7}$$

the differential equation (A.3.6) is rewritten as

$$\chi = 2s\gamma^3\left(\frac{1}{2}\cos\theta - \sin\theta\partial_\theta\right)\chi, \tag{A.3.8}$$

which is solved by

$$\chi_s = \begin{pmatrix} A\left(\sin\frac{\theta}{2}\right)^{\frac{1}{2}(1-s)}\cdot\left(\cos\frac{\theta}{2}\right)^{\frac{1}{2}(1+s)} \\ B\left(\sin\frac{\theta}{2}\right)^{\frac{1}{2}(1+s)}\cdot\left(\cos\frac{\theta}{2}\right)^{\frac{1}{2}(1-s)} \end{pmatrix}. \tag{A.3.9}$$

Thus the solution to (A.3.3),(A.3.4) is given by a linear combination of

$$\hat{\xi}_+ = e^{\frac{i}{2}\varphi}\rho(\tau)\begin{pmatrix} A_{(+)}\cos\frac{\theta}{2} \\ B_{(+)}\sin\frac{\theta}{2}\end{pmatrix}, \qquad \hat{\xi}_- = e^{-\frac{i}{2}\varphi}\rho(\tau)\begin{pmatrix} A_{(-)}\sin\frac{\theta}{2} \\ B_{(-)}\cos\frac{\theta}{2}\end{pmatrix}. \tag{A.3.10}$$

Let turn to the τ component of the Killing spinor equation (A.3.5). It does not have nontrivial solution for $V_\tau = 0$.[15] Turning on non-zero V_τ and setting $\rho =$ const., one finds that $\hat{\xi}_+$ satisfies (A.3.5) if

$$V_\tau \begin{pmatrix} A_{(+)} \\ B_{(+)} \end{pmatrix} = \frac{i}{2\tilde{\ell}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{(+)} \\ B_{(+)} \end{pmatrix}, \tag{A.3.11}$$

which is solved by $V_\tau = \pm\frac{i}{2\tilde{\ell}}$ and $B_{(+)} = \pm A_{(+)}$. In the same way, for $\hat{\xi}_-$ one finds

$$V_\tau \begin{pmatrix} A_{(-)} \\ B_{(-)} \end{pmatrix} = -\frac{i}{2\tilde{\ell}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} A_{(-)} \\ B_{(-)} \end{pmatrix}, \tag{A.3.12}$$

which is solved by $V_\tau = \mp\frac{i}{2\tilde{\ell}}$ and $B_{(-)} = \pm A_{(-)}$. The Killing spinor equation for $\bar{\xi}$ can be solved in the same way by noticing that only the sign in front of V_τ is flipped. On $S^2 \times S^1$ there are two pairs of Killing spinors:

$$\xi = e^{\frac{i}{2}\varphi} \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \qquad \bar{\xi} = e^{-\frac{i}{2}\varphi} \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}, \qquad \text{for } V_\tau = \frac{i}{2\tilde{\ell}}, \tag{A.3.13}$$

$$\xi' = e^{\frac{i}{2}\varphi} \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix}, \qquad \bar{\xi}' = e^{-\frac{i}{2}\varphi} \begin{pmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}, \qquad \text{for } V_\tau = -\frac{i}{2\tilde{\ell}}, \tag{A.3.14}$$

which are normalized, so that $\bar{\xi}\xi = \bar{\xi}'\xi' = -\cos\theta$. Note that (A.3.5) implies $H - i\not{K} = -2\gamma^3 V_\tau$, so the Killing spinors $\xi, \bar{\xi}$ are for $H = 0$, $\not{K} = \frac{1}{\tilde{\ell}}\gamma^3$, whereas $\xi', \bar{\xi}'$ are for $H = 0$, $\not{K} = -\frac{1}{\tilde{\ell}}\gamma^3$. If the pair $\xi, \bar{\xi}$ is chosen as the SUSY-preserving Killing spinors, the Killing vector is

$$\bar{\xi}\gamma^a\xi = (0, i\sin\theta, -1), \quad v = -\frac{1}{\tilde{\ell}}\partial_\varphi - i\partial_\tau. \tag{A.3.15}$$

[15] Suppose $\rho(\tau) = e^{\frac{i}{2}q\tau}$ with $q \in \mathbb{R}$, the differential equation of τ component (A.3.5):

$$\partial_\theta \chi = \frac{q\ell}{2}\gamma^2\chi$$

has no solution except $A = B = 0$.

Appendix B

Geometric Quantization

B.1 Prequantization

Let M be a symplectic manifold with a symplectic form $\omega = \frac{1}{2}\omega_{mn}\mathrm{d}x^m \mathrm{d}x^n$. Then for every vector field $X = X^m \partial_m$ such that $\pounds_X \omega = 0$, there is a function f called moment map satisfying

$$\mathrm{d}f + \imath_X \omega = 0, \tag{B.1.1}$$

or in components

$$\partial_n f + X^m \omega_{mn} = 0. \tag{B.1.2}$$

In what follows we denote the vector field corresponding to a function f by $X(f)$.

The prequantization is defined as the following map from functions $(f, g, h, \ldots)$ on M to differential operators $(\hat{f}, \hat{g}, \hat{h}, \ldots)$

$$\hat{f} \equiv -i\hbar X(f) - \imath_{X(f)}\vartheta + f, \tag{B.1.3}$$

acting on certain Hilbert space of wave functions. Here ϑ is a one-form satisfying $\omega = \mathrm{d}\vartheta$. One can show that, under this map, the classical Poisson bracket $\{f, g\} \equiv (\omega^{-1})^{mn}\partial_m f\, \partial_n g = h$ turns into the commutation relation:

$$[\,\hat{f}, \hat{g}\,] = i\hbar\, \hat{h}. \tag{B.1.4}$$

In order to show the above statement, it is enough to derive

$$[X(f), X(g)] = -X(h), \tag{B.1.5}$$

$$-i\hbar X(f)\{g - \imath_{X(g)}\vartheta\} + i\hbar X(g)\{f - \imath_{X(f)}\vartheta\} = i\hbar\{h - \imath_{X(h)}\vartheta\}. \tag{B.1.6}$$

First, note that the Poisson bracket can be written in terms of $X(f)$ by using (B.1.2).

$$\{f\,,\, g\} = -X(f) \cdot g \tag{B.1.7}$$

We thus obtain, for any function ϕ on M,

$$\begin{aligned}[X(f)\,,\,X(g)]\phi &= X(f)X(g)\cdot\phi - X(g)X(f)\cdot\phi \\ &= \{f,\{g,\phi\}\} - \{g,\{f,\phi\}\} \\ &= \{\{f,g\},\phi\} \\ &= -X(h)\cdot\phi.\end{aligned} \tag{B.1.8}$$

thereby proving (B.1.5). Here we used the Jacobi identity at the third equality. Second, the LHS of (B.1.6) is expanded as

$$\frac{\text{(LHS)}}{i\hbar} = 2h + X(f)^m X(g)^n(\partial_m\vartheta_n - \partial_n\vartheta_m) + (X(f)^m\partial_m X(g)^n - X(g)^m\partial_m X(f)^n)\vartheta_n. \tag{B.1.9}$$

The second term is $-\{f,g\} = -h$, and the third term is $\imath_{[X(f),X(g)]}\vartheta = -\imath_{X(h)}\vartheta$, thus (B.1.6) has been proved.

An Example: $M = \mathbb{R}^2$, $\omega = \mathrm{d}p\,\mathrm{d}q$ This example is a two-dimensional phase space which occur in classical mechanics for a single particle moving in one-dimension. The variables p, q stand for the position and the momentum. The vector fields which generate translations on M are given by

$$X(q) = -\partial_p, \qquad X(p) = \partial_q. \tag{B.1.10}$$

Here we used $w_{pq} = (w^{-1})^{qp} = 1$. Then we implement the prequantization as discussed above and obtain

$$\begin{aligned}\hat{q} &= -i\hbar X(q) - \imath_{X(q)}\vartheta + q = i\hbar\partial_p + q, \\ \hat{p} &= -i\hbar X(p) - \imath_{X(p)}\vartheta + p = -i\hbar\partial_q,\end{aligned} \tag{B.1.11}$$

where we set $\vartheta = p\mathrm{d}q$. These are familiar results of one-dimensional quantum mechanics except the term $i\hbar\partial_p$ appearing in $\hat{q}$. The polarization, the second step of the geometric quantization, settles this issue by restricting the quantum wavefunctions to depend only on q.

B.2 Complex structure

A manifold M is said to have a *complex structure* if M has an *almost complex structure* which is *integrable*. In that case, M can be covered by complex coordinate patches. An almost complex structure $J^m_{\ n}$ on M is a tensor field satisfying $J^2 = -1$. Since a real matrix J satisfying $J^2 = -1$ must have eigenvalues $\pm i$ with equal multiplicities, M must be a real even-dimensional manifold. The complexified tangent space $T_xM^{\mathbb{C}}$ can be decomposed into the eigenspace $T_xM^{\pm}$ of eigenvalues $J(x) = \pm i$.

$$T_xM^{\mathbb{C}} = T_xM^{+} \oplus T_xM^{-}. \tag{B.2.1}$$

Vector fields V ($\bar{V}$) are called holomorphic(anti-holomorphic) if $JV = +iV$ ($J\bar{V} = -i\bar{V}$), where the action of J on vector fields is defined as follows.

$$J:\, V = V^i\partial_i \longmapsto JV = V^i J^j_{\ i}\partial_j. \tag{B.2.2}$$

As an example, let us consider the tangent space of a complex manifold M of $\dim_{\mathbb{C}} M = k$. The tangent space T_pM is spanned by $2k$ vectors

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^k}\, ;\, \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, \ldots, \frac{\partial}{\partial y^k} \right\}. \tag{B.2.3}$$

With the same coordinates, T_p^*M is spanned by

$$\left\{ \mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^k\, ;\, \mathrm{d}y^1, \mathrm{d}y^2, \ldots, \mathrm{d}y^k \right\}. \tag{B.2.4}$$

Suppose that the coordinate x_m, y_m are chosen such a way that $J(p)$ satisfies

$$J(p)\left(\frac{\partial}{\partial x^m}\right) = \frac{\partial}{\partial y^m}, \qquad J(p)\left(\frac{\partial}{\partial y^m}\right) = -\frac{\partial}{\partial x^m}, \tag{B.2.5}$$

so that $J(p)^2 = -\mathrm{id}_{T_pM}$. Then by defining a set of $2k$ vectors by

$$\begin{aligned} \frac{\partial}{\partial z^m} &\equiv \frac{1}{2}\left(\frac{\partial}{\partial x^m} - i\frac{\partial}{\partial y^m}\right), \\ \frac{\partial}{\partial \bar{z}^m} &\equiv \frac{1}{2}\left(\frac{\partial}{\partial x^m} + i\frac{\partial}{\partial y^m}\right), \end{aligned} \tag{B.2.6}$$

one can check that the almost complex structure $J(p)$ acts on the complex basis (B.2.6) as

$$J(p)\left(\frac{\partial}{\partial z^m}\right) = i\frac{\partial}{\partial z^m}\,, \qquad J(p)\left(\frac{\partial}{\partial \bar{z}^m}\right) = -i\frac{\partial}{\partial \bar{z}^m}\,. \tag{B.2.7}$$

Then $z^m = x^m + iy^m$ is the complex coordinate on M. In fact the action of $J(p)$ is independent of the chart. Let $z^m = x^m + iy^m$ and $\omega^m = u^m + iv^m$ be two charts overlapping at p. As the function $z^m = z^m(\omega)$ satisfy the Cauchy-Riemann relations on this overlap, one finds

$$J(p)\left(\frac{\partial}{\partial u^m}\right) = J(p)\left(\frac{\partial x^n}{\partial u^m}\frac{\partial}{\partial x^n} + \frac{\partial y^n}{\partial u^m}\frac{\partial}{\partial y^n}\right) = \frac{\partial}{\partial v^m}, \tag{B.2.8}$$

and likewise,

$$J(p)\left(\frac{\partial}{\partial v^m}\right) = -\frac{\partial}{\partial u^m}. \tag{B.2.9}$$

Therefore, $J(p)$ takes the form

$$J(p) = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}, \tag{B.2.10}$$

with respect to the coordinates x^m, y^m as well as u^m, v^m, where I_k is the $k \times k$ unit matrix.

An almost complex structure J is integrable if the Nijenhuis tensor N^i_{jk} defined by

$$N^k_{ij} \equiv J^{\,l}{}_i(\partial_l J^k{}_i - \partial_j J^k{}_l) - J^{\,l}{}_j(\partial_l J^k{}_i - \partial_i J^k{}_l) \tag{B.2.11}$$

vanishes. This condition is actually equivalent to that the Lie bracket of two holomorphic vector fields be holomorphic. That is to say, for arbitrary vector fields V, W, the following equation holds:

$$(J - i)[(J + i)V, (J + i)W] = 0 \tag{B.2.12}$$

Using $J^2 = -1$ and then $i(J-i) = -J(J-i)$, the LHS of (B.2.12) can be written as

$$\text{LHS} = (J-i)\left\{[JX, JV] - [V, W] - J[JX, V] - J[X, JV]\right\}. \tag{B.2.13}$$

The expression inside the big parenthesis above is rewritten in component notation.

$$\begin{aligned}
\{\cdots\} &= (JV)^i\partial_i(JW)^k - (JW)^i\partial_i(JV)^k - V^i\partial_iW^k + W^i\partial_iV^k \\
&\quad - J^k_{\ l}\left\{(JV)^i\partial_iW^l - (JW)^i\partial_iV^l + V^i\partial_i(JW)^l - W^i\partial_i(JV)^l\right\} \\
&= (JV)^i\partial_iJ^k_{\ l}\cdot W^l - (JW)^i\partial_iJ^k_{\ l}\cdot V^l - J^k_{\ l}\left\{V^i\partial_iJ^l_{\ m}\cdot W^m - W^i\partial_iJ^l_{\ m}\cdot V^m\right\} \\
&= V^jW^l\left\{J^i_{\ j}\partial_iJ^k_{\ l} - J^i_{\ j}\partial_iJ^k_{\ l} - J^k_{\ l}\partial_jJ^i_{\ l} + J^k_{\ i}\partial_lJ^i_{\ j}\right\} \\
&= V^jW^lN^k_{jl}.
\end{aligned}$$

So the condition (B.2.11) is equivalent to $N^k_{ij} = 0$ as claimed.

Now let us make some comments on the case $M = \mathrm{Ad}_G(\lambda) = G/K$ in Section 3.2.1. If $\mathfrak{g} = \mathrm{Lie}(G)$ is decomposed as $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{n}$ such that $[\mathfrak{k},\mathfrak{n}]\subset\mathfrak{n}$, then $\mathfrak{n}$ is identified with $T_\lambda M$. An almost complex structure (B.2.1) thus corresponds to a decomposition

$$\mathfrak{n}^{\mathbb{C}} = \mathfrak{n}^+ \oplus \mathfrak{n}^-, \tag{B.2.14}$$

where $\mathfrak{n}^\pm$ are the eigenspaces of $J = \pm i$. This almost complex structure is integrable if $\mathfrak{n}^\pm$ are closed under Lie bracket relations, namely

$$[\mathfrak{n}^+,\mathfrak{n}^+]\subset\mathfrak{n}^+, \qquad [\mathfrak{n}^-,\mathfrak{n}^-]\subset\mathfrak{n}^-. \tag{B.2.15}$$

The adjoint orbit M then has a complex structure: in other words, there exist a system of local complex coordinate $(z^I, \bar{z}^{\bar{J}})$ on M.

Let $g(z^I, \bar{z}^{\bar{J}})$ be the map from M to G introduced in Section 3.2, expressed as a function of the complex coordinate. The the tangent space of M at $(z^I, \bar{z}^{\bar{J}})$ is given by

$$T^{\mathbb{C}}_{(z^I,\bar{z}^{\bar{J}})}M = \mathrm{Ad}_{g(z^I,\bar{z}^{\bar{J}})}(\mathfrak{n}^{\mathbb{C}}) = \mathrm{Ad}_{g(z^I,\bar{z}^{\bar{J}})}(\mathfrak{n}^+)\oplus\mathrm{Ad}_{g(z^I,\bar{z}^{\bar{J}})}(\mathfrak{n}^-), \tag{B.2.16}$$

where $\mathrm{Ad}_{g(z^I,\bar{z}^{\bar{J}})}(\mathfrak{n}^{\mathbb{C}}) \equiv \{g\xi g^{-1} | \xi\in\mathfrak{n}^{\mathbb{C}}\}$. This implies a relation of the form Eq. (3.2.6) between the holomorphic tangent vectors $\frac{\partial}{\partial z^I}$ and the elements of $\mathrm{Ad}_{g(z^I,\bar{z}^{\bar{J}})}(\mathfrak{n}^+)$.

$$\begin{aligned}
\frac{\partial g(z^I,\bar{z}^{\bar{J}})}{\partial z^J} &= \xi^J\cdot g(z^I,\bar{z}^{\bar{J}}) + g(z^I,\bar{z}^{\bar{J}})\cdot\eta^J &\qquad& (\xi^J\in\mathrm{Ad}_{g(z^I,\bar{z}^{\bar{J}})}(\mathfrak{n}^+)) \\
&= g(z^I,\bar{z}^{\bar{J}})(\tilde{\xi}^J + \eta^J) && (\tilde{\xi}^J\in\mathfrak{n}^+)
\end{aligned}$$

and similarly for the antiholomorphic counterpart. Equivalently, if we denote $\mathrm{d} = \partial + \bar{\partial}$, one has

$$g^{-1}\partial g\in\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{n}^+, \qquad g^{-1}\bar{\partial}g\in\mathfrak{k}^{\mathbb{C}}\oplus\mathfrak{n}^-. \tag{B.2.17}$$

Then we find that KKS 2-form (3.2.7) is of type $(1,1)$ with respect to this complex structure.

$$\begin{aligned}\omega &= -2i\,\mathrm{Tr}[\lambda\left((g^{-1}\partial g)^2 + (g^{-1}\bar{\partial} g)^2 + g^{-1}\partial g\cdot g^{-1}\bar{\partial} g + g^{-1}\bar{\partial} g\cdot g^{-1}\partial g\right)] \\ &= -2i\,\mathrm{Tr}[\lambda(\partial g^{-1}\cdot\bar{\partial} g + \bar{\partial} g^{-1}\cdot\partial g)]\end{aligned}$$

Here the first and second terms of first line vanish due to (B.2.17). Let $g^{-1}\partial g = \mu_i H_i + \mu_\alpha E_\alpha$, where $H_i \in \mathfrak{k}$, $E_\alpha \in \mathfrak{n}^+$ and μ_i, μ_α are one-forms. Then the first term can be rewritten as follows.

$$\begin{aligned}\mathrm{Tr}[\lambda(g^{-1}\partial g)^2] &= \mathrm{Tr}\left[\lambda(\mu_i H_i + \mu_\alpha E_\alpha)^2\right] \\ &= \mathrm{Tr}\left[\lambda([H_i, H_j]\mu_i\mu_j + [E_\alpha, E_\beta]\mu_\alpha\mu_\beta + (H_i E_\alpha - E_\alpha H_i)\mu_i\mu_\alpha)\right]\end{aligned}$$

One can show each term is indeed zero by using the fact that $\mathrm{Tr}[AB]$ is nonzero only if $A \in \mathfrak{n}^\pm$, $B \in \mathfrak{n}^\mp$ or $A, B \in \mathfrak{k}$. Likewise, the second term $\lambda(g^{-1}\bar{\partial} g)^2$ is zero. Thus the KKS symplectic form is of type $(1,1)$, so M is Kähler manifold.

Appendix C

Jeffrey-Kirwan residue

C.1 Basic idea

Here we review the basic idea of Jeffrey-Kirwan(JK) residue to understand the detail of the computation of indices for $\mathcal{N}=2$ SQMs discussed in Section 4.2. We mainly follow [50].

Let us consider the Gaussian path integral of a chiral multiplet (4.1.3) in a representation $\Lambda_{\rm c}$ of a gauge group G:

$$L_{\rm c} = (\bar{\phi}\,,\,\bar{\psi}) \begin{pmatrix} -D_t^2+\sigma^2-iD & -i\lambda \\ -i\bar{\lambda} & -iD_t-i\sigma \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}.$$

We regard the vectormultiplet fields $A_t, \sigma, \lambda, \bar{\lambda}, D$ here as Cartan valued constants. The one-loop determinant for the chiral multiplet is then given by

$$\begin{aligned} &\prod_{n\in\mathbb{Z}} \mathrm{sdet} \begin{pmatrix} (n-u)(n-\bar{u})-iD & -i\lambda \\ -i\bar{\lambda} & n-\bar{u} \end{pmatrix} \\ &\qquad = \prod_{n\in\mathbb{Z}} \prod_{q_i\in\Lambda_{\rm c}} \frac{q_{ia}\bar{u}^a - n}{|q_{ia}u^a-n|^2 - iq_{ia}D^a - (q_{ia}u^a-n)^{-1}q_{ia}\lambda^a q_{ia}\lambda^a}, \end{aligned} \tag{C.1.1}$$

where q_i is the charge of the i-th chiral multiplet component and $a=1,2,\ldots,r=\mathrm{rank}(G)$. We rewrite this infinite product further by introducing the fermions ξ,η defined by

$$\xi \equiv i(\epsilon\bar{\lambda}+\bar{\epsilon}\lambda), \qquad \eta \equiv \frac{1}{2}(\epsilon\bar{\lambda}-\bar{\epsilon}\lambda).$$

These are linear combinations of $\lambda, \bar{\lambda}$ such that $\xi\eta=\lambda\bar{\lambda}$ and $\boldsymbol{Q}$ acts on the vectormultiplet variables as

$$\boldsymbol{Q}u=0, \qquad \boldsymbol{Q}\bar{u}=\xi, \qquad \boldsymbol{Q}\xi=0, \qquad \boldsymbol{Q}\eta=D, \qquad \boldsymbol{Q}D=0. \tag{C.1.2}$$

We denote the linear function of u^a that appear in the denominator of (C.1.1) collectively as u_i, so that

$$\prod_{n\in\mathbb{Z}} \prod_{q_i\in\Lambda_{\rm c}} (q_{ia}u^a-n) = \prod_i u_i.$$

Similar notation will be used for their complex conjugate $\bar{u}$ and the other variables D, ξ, η as well. The infinite product (C.1.1) becomes

$$\prod_i \frac{\bar{u}_i}{|u_i|^2 - iD_i + i\xi_i\eta_i(\bar{u}_i)^{-1}}, \tag{C.1.3}$$

thus the integral of our interest for rank-r gauge theory is written as follows.

$$\int \mathrm{d}^r u \mathrm{d}^r \bar{u} \mathrm{d}^r \xi \mathrm{d}^r \eta \mathrm{d}^r D \cdot e^{-\frac{D^2}{2e^2} - i\zeta D} \prod_i \frac{\bar{u}_i}{|u_i|^2 - iD_i + i\xi_i\eta_i(\bar{u}_i)^{-1}} \cdot (\cdots). \tag{C.1.4}$$

Here $(\cdots)$ is the contribution to the determinant form vector and fermi multiplets.

In order to avoid problems of divergence we will encounter later, at this point we remove the tubular neighborhood of $u_i = 0$ for each i from the u-integration domain. Also, we shift the D-integration contour off the real axis so that $D^a \in \mathbb{R} - i\delta^a \; (a = 1, 2, \ldots, r)$.

Rank-1 case. For simplicity we first consider the rank-1 case, assuming that u_i is the following linear function of u.

$$u_i = q_i u - n_i$$

For simplicity, we focus on the jth factor of infinite product (C.1.4) and consider whether to pick the residue of the pole $u_i = 0 \, (u = in_i/q_i)$. Expanding it with respect to $\xi_i\eta_i$,

$$\frac{\bar{u}_i}{|u_i|^2 - iD_i} - \frac{i\xi_i\eta_i}{(|u_i|^2 - iD_i)^2} + \cdots ,$$

one can integrate these fermions out, and the integral we want to evaluate is then,

$$\int \mathrm{d}u \mathrm{d}\bar{u} \mathrm{d}\xi \mathrm{d}\eta \mathrm{d}D \, \frac{\bar{u}_i}{|u_i|^2 - iD_i + i\xi_i\eta_i(\bar{u}_i)^{-1}} = \frac{1}{|q_i|} \int_M \frac{\mathrm{d}u_i \mathrm{d}\bar{u}_i \mathrm{d}D_i}{i(|u_i|^2 - iD_i)} \tag{C.1.5}$$

$$= \frac{1}{|q_i|} \int_{\mathbb{R} - iq_i\delta} \frac{\mathrm{d}D_i}{D_i} \int_M \mathrm{d}u_i \mathrm{d}\bar{u}_i \bar{\partial}_i \left(\frac{\bar{u}_i}{|u_i|^2 - iD_i} \right)$$

$$= \frac{1}{|q_i|} \int_{\mathbb{R} - iq_i\delta} \frac{\mathrm{d}D_i}{D_i} \int_{\partial M} \frac{\bar{u}_i \mathrm{d}u_i}{|u_i|^2 - iD_i} \tag{C.1.6}$$

Here M denote the complex u_i-plane with the origin removed. In the third equality, we applied the Stokes theorem to the u_h-integral, and then rewrote it as a contour integral along ∂M. Also, the contour of D_i-integral is deformed from the real axis to $\mathbb{R} - iq_i\delta$. There is a simple pole at $D_i = 0$ in (C.1.6), even though $D_i = 0$ was perfectly regular in (C.1.5). That is why we deformed D_i contour in advance. Now we evaluate the integral above in each components of ∂M. First, we focus on one of ∂M, the one going around the hole. Another one at infinity will be discussed later.

Whether the boundary component contributes to the integral depends on the sign of $q_i\delta$.

- If $q_i\delta < 0$, then $\mathrm{Re}(|u_i|^2 - iD_i) = |u_i|^2 - q_i\delta > 0$. The radius of the u_i contour can be shrunk as one likes. In this case, there is no contribution to the integral from contour around $u_i = 0$.

- If $q_i\delta > 0$, then $\mathrm{Re}(|u_i|^2 - iD_i)$ can vanish. To evaluate this contribution, we deform the D_i contour that is below the real axis to be above it. In this deformation, it hits the pole at $D_i = 0$, then we find that one should integrate over the D_i contour along a line above the real axis plus a circle going around the origin counterclockwise. Similarly to the previous case, when D_i takes value on a line above the real axis, the u_i contour can be shrunk without any problem. The D_i contour over this line thus contributes nothing. The contribution from the circle can be evaluated easily, and leaves us with a u_i-contour integral that goes around the pole at $u_i = 0$ clockwise. One obtains,

$$\frac{1}{|q_i|}\oint_{\mathrm{ccw}}\frac{\mathrm{d}D_i}{D_i}\oint_{\mathrm{cw}}\frac{\bar{u}_i\mathrm{d}u_i}{|u_i|^2 - iD_i} = \frac{2\pi i}{|q_i|}\oint_{\mathrm{cw}}\frac{\mathrm{d}u_i}{u_i} = \frac{4\pi^2}{|q_i|}. \tag{C.1.7}$$

In summary, the whole contribution of the bulk poles are expressed as follows.

$$I_{\mathrm{bulk}} = \begin{cases} \displaystyle\sum_{i(q_i>0)}\mathrm{Res}_{u=n_i/q_i}\left(\frac{\mathrm{d}u}{\prod_i(q_iu - n_i)}\right) & (\delta > 0) \\ \displaystyle -\sum_{i(q_i<0)}\mathrm{Res}_{u=n_i/q_i}\left(\frac{\mathrm{d}u}{\prod_i(q_iu - n_i)}\right) & (\delta < 0) \end{cases}. \tag{C.1.8}$$

Let us next consider the contribution of the boundary of ∂M at infinity. In this case, since the FI parameter ζ plays significant role, we study the following integral with ζ restored.

$$\begin{aligned} I &\equiv \frac{1}{4\pi^2}\int \mathrm{d}u\mathrm{d}\bar{u}\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}D\, e^{-\frac{D^2}{2e^2} - i\zeta D}\prod_i \frac{\bar{u}_i}{|u_i|^2 - iD_i + i\xi_i\eta_i/\bar{u}_i} \\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}-i\delta}\frac{\mathrm{d}D}{D}\, e^{-\frac{D^2}{2e^2} - i\zeta D}\int_{\partial M}\mathrm{d}u\prod_i\frac{q_i\bar{u} - n_i}{|q_iu - n_i|^2 - iq_iD_i} \end{aligned}$$

When u_i are on the circle at infinity, one may integrate over D first. If $\zeta > 0$ (or $\zeta < 0$), one can close the D-integration contour in the lower (or upper) half-plane. Besides $D = 0$, there seem to be many poles at $D = |q_iu - n_i|^2/iq_i$. But it was argued in [18] that the limit $e \to 0$ should be taken with $\zeta' \equiv \zeta e^2$ and $D' \equiv D/e^2$ held fixed. So all the poles of the integrand except $D' = 0$ are at a distance $\sim e^{-2}$ from the origin and negligible. As a result, the D-contour contains only the pole at $D = 0$ for $\zeta\delta < 0$. After the D-integral one obtains a u-contour integral which picks up all the bulk pole residues.

$$I_\infty = \begin{cases} \displaystyle -\sum_{\text{all } i}\mathrm{Res}_{u=n_i/q_i}\left(\frac{\mathrm{d}u}{\prod_i(q_iu - n_i)}\right) & (\delta > 0,\ \zeta < 0) \\ \displaystyle \sum_{\text{all } i}\mathrm{Res}_{u=n_i/q_i}\left(\frac{\mathrm{d}u}{\prod_i(q_iu - n_i)}\right) & (\delta < 0,\ \zeta > 0) \\ 0 & (\text{otherwise}) \end{cases}, \tag{C.1.9}$$

The sum of the bulk pole and infinity contribution depends on the sign of ζ but not on the regularization parameter δ.

- $\zeta > 0$

$$I \equiv I_{bulk} + I_{\infty} = \begin{cases} \displaystyle\sum_{i(q_i>0)} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right) + 0 & (\delta > 0) \\ \displaystyle -\sum_{i(q_i<0)} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right) + \sum_{\text{all } i} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right) & (\delta < 0) \end{cases}$$

$$= \sum_{i(q_i>0)} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right). \tag{C.1.10}$$

- $\zeta < 0$

$$I \equiv I_{bulk} + I_{\infty} = \begin{cases} \displaystyle\sum_{i(q_i>0)} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right) - \sum_{\text{all } i} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right) & (\delta > 0) \\ \displaystyle -\sum_{i(q_i<0)} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right) + 0 & (\delta < 0) \end{cases}$$

$$= -\sum_{i(q_i<0)} \mathrm{Res}_{u=n_i/q_i} \left(\frac{\mathrm{d}u}{\prod_i (q_i u - n_i)} \right). \tag{C.1.11}$$

So it is convenient to choose $\delta = \zeta$ so that one can ignore the contribution from infinity. Then we should take poles at $u_i = 0$ such that $q_i \zeta > 0$ only.

Rank-r case. For the case with general rank r, one has

$$\begin{gathered} u_i = n_i - i q_{ia} u_a, \qquad \bar{u}_i = n_i + i q_{ia} \bar{u}_a, \\ \xi_i = q_{ia} \zeta_a, \qquad \eta_i = q_{ia} \eta_a, \qquad D_i = q_{ia} D_a, \end{gathered} \tag{C.1.12}$$

where $a = 1, \ldots, r$. Let us first consider whether or not the intersection of r hyperplanes

$$u_1 = u_2 = \cdots = u_r = 0 \tag{C.1.13}$$

contributes to the integral for simplicity. As in the previous analysis, we extract only the relevant factors from the full integrand.

$$\begin{aligned} I &= \int \prod_{a=1}^{r} \mathrm{d}u_a \mathrm{d}\bar{u}_a \mathrm{d}\xi_a \mathrm{d}\eta_a \mathrm{d}D_a \prod_{i}^{r} \frac{\bar{u}_i}{|u_i|^2 - iD_i + \xi_i \eta_i (\bar{u}_i)^{-1}} \\ &= \frac{1}{|\det(q_{ia})|} \int \mathrm{d}^r u \prod_{i=1}^{r} \frac{\mathrm{d}\bar{u}_i \mathrm{d}D_i}{i(|u_i|^2 - iD_i)^2}. \end{aligned} \tag{C.1.14}$$

Since this is the product of r copies of the rank-1 problem (C.1.6), one may well think that this pole contributes to the integral if $q_{ai}\delta_a > 0$ for all i. However, this is not the correct JK prescription for a general rank-r. Nevertheless, let us continue the discussion based on this regularization for a while.

In the rank-1 case, a d^2u integral was transformed into a contour $\mathrm{d}u$-integral using Stokes theorem. Generalization of this to rank-r case a little involved. Let us first consider the integral (C.1.14) with the domain of u-integration

$$M \equiv \{(u_1, u_2, \ldots, u_r) \in \mathbb{C}^r \big| \, |u_i| \geq \epsilon \text{ for all } i\}. \tag{C.1.15}$$

The boundary of M and that of ∂M are given by

$$\begin{aligned} \partial M &= -\sum_i S_i, \qquad S_i \equiv \{|u_i| = \epsilon;\ |u_j| \leq \epsilon \text{ for all other } j\}, \\ \partial S_i &= -\sum_{j\neq i} S_{ij}, \qquad S_{ij} \equiv \{|u_i| = |u_j| = \epsilon;\ |u_k| \leq \epsilon \text{ for all other } k\} = S_i \cap S_j, \\ &\ \vdots \end{aligned} \tag{C.1.16}$$

where we ignored the boundary at infinity for simplicity. Moreover, there is a cell decomposition of M of the form

$$M = \sum_i C_i, \quad \partial C_i = \sum_{j\neq i} C_{ij} - S_i, \quad \partial C_{ij} = \sum_{k\neq i,j} C_{ijk} - S_{ij}, \ \ldots. \tag{C.1.17}$$

The integral we consider (C.1.14) is rewritten using a $3r$-form μ,

$$I = \frac{1}{|\det(q_{ia})|}\int_{M\times\Gamma} \mu, \qquad \mu \equiv \prod_{i=1}^r \frac{\mathrm{d}\bar{u}_i \mathrm{d}D_i}{i(|u_i|^2 - iD_i)^2} \wedge \mathrm{d}^r u, \tag{C.1.18}$$

where Γ is the D-contour. The exterior derivative with respect to $\bar{u}_i$: $\bar{\partial} \equiv \mathrm{d}u_i \frac{\partial}{\partial \bar{u}_i}$ raises the degree of differential form by one. As an example one has

$$\bar{\partial}\mu_i = \mu, \tag{C.1.19}$$

where the $(3r-1)$-form μ_i is given by

$$\mu_i = \frac{\bar{u}_i \mathrm{d}D_i}{D_i(|u_i|^2 - iD_i)} \wedge \prod_{j\neq i}^r \frac{\mathrm{d}\bar{u}_j \mathrm{d}D_j}{i(|u_j|^2 - iD_j)^2} \wedge \mathrm{d}^r u. \tag{C.1.20}$$

More generally, one can define the $(3r-p)$-form satisfying the decent relation as follows.

$$\begin{aligned} \mu_{i_1 i_2\cdots i_p} &= \prod_{j\in\{i_1,i_2,\ldots,i_p\}} \frac{\bar{u}_i \mathrm{d}D_i}{D_i(|u_i|^2 - iD_i)} \prod_{j\notin\{i_1,i_2,\ldots,i_p\}}^r \frac{\mathrm{d}\bar{u}_j \mathrm{d}D_j}{i(|u_j|^2 - iD_j)^2} \wedge \mathrm{d}^r u, \\ \bar{\partial}\mu_{i_1 i_2\cdots i_p} &= \mu_{i_2 i_3\cdots i_p} - \mu_{i_1 i_3\cdots i_p} + \cdots + (-)^{p-1}\mu_{i_1 i_2\cdots i_{p-1}} \end{aligned} \tag{C.1.21}$$

By applying Stokes theorem to the integral (C.1.18) once, one obtains

$$\begin{aligned} \int_M \mu &= \sum_i \int_{C_i} \bar{\partial}\mu_i \\ &= \sum_i \left(\sum_{j\neq i} \int_{C_{ij}} \mu_i - \int_{S_i} \mu_i \right) \\ &= \sum_{i<j} \int_{C_{ij}} (\mu_i - \mu_j) - \sum_i \int_{S_i} \mu_i. \end{aligned} \tag{C.1.22}$$

In the second equality, we used the cell decomposition (C.1.17). The first term on the last line is the integral over a cell C_{ij} of the differential form $\bar{\partial}\mu_{ij}$. By r times repeated use of Stokes theorem, the result is as follows.

$$\begin{aligned}
\int_M \mu = \cdots = & \sum_{i_1<i_2<\cdots<i_r} \left(\sum_{j\neq i_1,i_2,\ldots,i_r} \int_{C_{i_1 i_2\cdots i_r}} \mu_{i_1 i_2\cdots i_r} - \int_{S_{i_1 i_2\cdots i_r}} \mu_{i_1 i_2\cdots i_r} \right) \\
& - \sum_{p=1}^{r-1} \sum_{i_1<i_2<\cdots<i_p} \int_{S_{i_1 i_2\cdots i_p}} \mu_{i_1 i_2\cdots i_p} \\
= & - \sum_{p=1}^{r} \sum_{i_1<i_2<\cdots<i_p} \int_{S_{i_1 i_2\cdots i_p}} \mu_{i_1 i_2\cdots i_p}
\end{aligned} \tag{C.1.23}$$

Let us now integrate also with respect to D_i over $\Gamma = \{D_i \in \mathbb{R} - i\delta_i\}$, where $\delta_i \equiv q_{ia}\delta_a$. We take the D-integral into account and consider an integral, a part of (C.1.23).

$$\int_{S_{i_1 i_2\cdots i_p}\times\Gamma} \mu_{i_1 i_2\cdots i_p} \tag{C.1.24}$$

The integral vanishes if $\delta_j > 0$ for some $j \in \{i_1, i_2, \ldots, i_p\}$, because $S_{i_1 i_2\cdots i_p}$ can then be shrunk without problem. If $\delta_j > 0$ for all $j \in \{i_{1,i_2,\ldots,i_p}\}$, then we move the integration contour for $D_{i_1}, D_{i_2}, \ldots, D_{i_p}$ to the other side of the origin as in the rank-1 case. We thus find

$$\begin{aligned}
I \equiv \int_{M\times\Gamma} \mu = & - \sum_{p=1}^{r} \sum_{i_1<i_2<\cdots<i_p} \int_{S_{i_1 i_2\cdots i_p}\times\Gamma} \mu_{i_1 i_2\cdots i_p} \\
= & - \sum_{p=1}^{r} \sum_{i_1<i_2<\cdots<i_p} \prod_{j\in\{i_1,i_2,\ldots,i_p\}} \Theta(\delta_j)\cdot I_{i_1 i_2\cdots i_p}
\end{aligned} \tag{C.1.25}$$

where $\Theta(x)$ is the step function, equal to one if $x > 0$ and zero otherwise, and

$$I_{i_1 i_2\cdots i_p} \equiv \int_{S_{i_1 i_2\cdots i_p}\times\Gamma_{i_1 i_2\cdots i_p}} \mu_{i_1 i_2\cdots i_p}, \tag{C.1.26}$$

$$\Gamma_{i_1 i_2\cdots i_p} \equiv S^1_{D_{i_1}} \times S^1_{D_{i_2}} \times \cdots S^1_{D_{i_p}} \times \prod_{j\notin\{i_1,i_2,\ldots,i_p\}} (\mathbb{R} - i\delta_j). \tag{C.1.27}$$

Note that $I_{i_1 i_2\cdots i_p}$ is a $(3r-p)$-dimensional integral, of which the integration with respect to $(u_1, u_2, \ldots, u_p;\ D_1, D_2, \ldots, D_p)$ can be performed straightforwardly. The resulting $3(r-p)$-dimensional integral is the rank-$(r-p)$ version of the same problem, so that by repeatedly applying Stokes theorem to it one obtains

$$\begin{aligned}
I_{i_1 i_2\cdots i_p} = & - \sum_{q=p+1}^{r} \sum_{i_{p+1}<\cdots<i_q} \int_{S_{i_1 i_2\cdots i_q}\times\Gamma_{i_1 i_2\cdots i_q}} \mu_{i_1 i_2\cdots i_q} \\
= & - \sum_{q=p+1}^{r} \sum_{i_{p+1}<\cdots<i_q} \prod_{j\in\{i_{p+1},\ldots,i_q\}} \Theta(\delta_j)\cdot I_{i_1 i_2\cdots i_q}.
\end{aligned} \tag{C.1.28}$$

The relation (C.1.25) can be thought of as the case $p = 0$ of the (C.1.28).

By using (C.1.28) repeatedly, one can prove the following formula by induction.

$$I_{i_1 i_2 \cdots i_p} = (-)^{r-p} \left[\prod_{j \notin \{i_1, i_2, \ldots, i_p\}} \Theta(\delta_j) \right] \cdot I_{i_1 i_2 \cdots i_r}, \tag{C.1.29}$$

This implies that, for the special case $p = 0$, the regularized integral

$$I = (-1)^r \left[\prod_{i=1}^{r} \Theta(\delta_i) \right] \cdot I_{i_1 i_2 \cdots i_r} \tag{C.1.30}$$

picks up the residue of the pole $u_1 = u_2 = \cdots = u_r = 0$ if $\delta_i = q_{ia}\delta_a > 0$ for all i.

Genelarization. The above argument can be generalized to the cases where the integrand consists of more than r factors so that the domain of u-integration has a collection of $k(> r)$ hyperplanes. One can show that the following differential forms

$$\mu_{i_1 i_2 \cdots i_p} = \prod_k \frac{\bar{u}_k}{|u_k|^2 - iD_k} \prod_{j \in \{i_1, i_2, \ldots, i_p\}} \frac{\mathrm{d}D_j}{D_j} \wedge \frac{1}{(r-p)!} \left(\sum_\ell \frac{\mathrm{d}\bar{u}_\ell \mathrm{d}D_\ell}{iu_\ell(|u_\ell|^2 - iD_\ell)} \right)^{r-p} \wedge \mathrm{d}^r u \tag{C.1.31}$$

satisfy the decent relation (C.1.21). The proof goes as follows. Let us first introduce the notation

$$\prod_k \frac{\bar{u}_k}{|u_k|^2 - iD_k} \equiv g, \qquad \sum_\ell \frac{\mathrm{d}\bar{u}_\ell \mathrm{d}D_\ell}{i\bar{u}_\ell(|u_\ell|^2 - iD_\ell)} = \mathrm{d}\bar{u}^a h_{ab} \mathrm{d}D^b = h_b \mathrm{d}D^b, \tag{C.1.32}$$

and then (C.1.31) is rewritten as

$$\mu_{i_1 i_2 \cdots i_p} = g \left[\prod_{j \in \{i_1, i_2, \ldots, i_p\}} \frac{\mathrm{d}D_j}{D_j} \right] \wedge \frac{1}{(r-p)!} (h_a \mathrm{d}D^a)^{r-p} \wedge \mathrm{d}^r u. \tag{C.1.33}$$

Using the relation

$$\bar{\partial} g \equiv \mathrm{d}\bar{u}^a \frac{\partial g}{\partial \bar{u}^a} = g \sum_\ell \frac{\mathrm{d}\bar{u}_\ell D_\ell}{i\bar{u}_\ell(|u_\ell|^2 - iD_\ell)} = g h_b D^b, \tag{C.1.34}$$

one can expand $\bar{\partial}\mu_{i_1 i_2 \cdots i_p}$ explicitly as follows.

$$\bar{\partial}\mu_{i_1 i_2 \cdots i_p} = g\,(h_b D^b) \frac{q_{i_1 a_1} \cdots q_{i_p a_p} \mathrm{d}D^{a_1} \cdots \mathrm{d}D^{a_p}}{D_{i_1} \cdots D_{i_p}} \wedge \frac{1}{(r-p)!} (h_{a_{p+1}} \mathrm{d}D^{a_{p+1}} \cdots h_{a_r} \mathrm{d}D^{a_r}) \wedge \mathrm{d}^r u \tag{C.1.35}$$

In the RHS, the indices $a_1 \cdots a_r$ must be all different and, since h_a's anticommute, the index b of $h_b D^b$ has to agree with one of $\{a_1, \ldots, a_p\}$. For example $b = a_1$,

$$\begin{aligned}
\mathrm{RHS}(b = a_1) &= g h_{a_1'} D^{a_1'} \cdot \frac{q_{i_1 a_1} \mathrm{d}D^{a_1}}{D_{i_1}} \cdot \frac{q_{i_2 a_2} \cdots q_{i_p a_p} \mathrm{d}D^{a_2} \cdots \mathrm{d}D^{a_p}}{D_{i_2} \cdots D_{i_p}} \wedge \frac{1}{(r-p)!} (h_a \mathrm{d}D^a)^{r-p} \wedge \mathrm{d}^r u \\
&= g \left[\prod_{j \in \{i_2, \ldots, i_p\}} \frac{\mathrm{d}D_j}{D_j} \right] \wedge \frac{1}{(r-p+1)!} (h_a \mathrm{d}D^a)^{r-(p-1)} \wedge \mathrm{d}^r u \\
&= \mu_{i_2 \cdots i_p}.
\end{aligned}$$

So the summation of the all b follows (C.1.21). As in the rank-1 case we remove from the domain M of u-integration the tubular neighborhood of the singular hyperplane $u_k = 0$ for each k. Denoting the surface of the k-th tube by S_k we have as (C.1.16),

$$\partial M = -\sum_k S_k, \qquad \partial S_k = -\sum_{l \neq k} S_{kl}, \cdots . \tag{C.1.36}$$

Moreover, M can be cell-decomposed into the form (C.1.17) by defining C_k as the set of points of M whose nearest boundary component is S_k. Then the repeated application of the Stokes' theorem works in the same way as the before.

C.2 The rule of JK-residue

One of the regularization prescription introduced above, namely the rule of the shift of the D-integration contour, needs to be reconsidered here. In (C.1.27) we defined $\Gamma_{i_1 \cdots i_p}$ as a product of a p-torus, $|q_j \cdot D| \equiv |q_{ja} D^a| = \epsilon$ for $j \in \{i_1, i_2, \ldots, i_p\}$, and $(r-p)$-dimensional hyperplane $\mathrm{Im}(q_j \cdot D) = -q_j \cdot \delta$ for $j \notin \{i_1, i_2, \ldots, i_p\}$. As long as one considers the problem for which the integrand has exactly r singular hyperplanes intersecting at a point (C.1.13), the definition of $\Gamma_{i_1 i_2 \cdots i_p}$ (C.1.27) dose not cause any problem. However, in general one needs to consider many poles if there are more than r hyperplanes as noted above.

In this general case, the imaginary shift of the contour $\Gamma_{i_1 \cdots i_p}$ should be specified by a vector $\delta_{i_1 i_2 \cdots i_p}$ such that

$$q_{i_1} \cdot \delta_{i_1 i_2 \cdots i_p} = \cdots = q_{i_p} \cdot \delta_{i_1 i_2 \cdots i_p} = 0. \tag{C.2.1}$$

Given a δ and $\{q_{i_1}, q_{i_2}, \ldots, q_{i_p}\}$, there is no way to determine such a $\delta_{i_1 i_2 \cdots i_p}$ uniquely. Therefore it seems that, as was explained in [50], the only thing we can do is to introduce an independent shift parameter $\delta_{i_1 i_2 \cdots i_p}$ for each of $\Gamma_{i_1 \cdots i_p}$.

The JK-residue prescription begins by choosing a reference vector $\eta \in \mathfrak{h}^*$. For each D-integration contour $\Gamma_{i_1 \cdots i_p}$, the shift parameter $\delta_{i_1 i_2 \cdots i_p}$ has to be chosen so that

$$\eta \cdot \delta_{i_1 i_2 \cdots i_p} > 0 \tag{C.2.2}$$

is satisfied in addition to (C.2.1). Then the rest of the computation is much the same as before. One obtains a set of recursion relations

$$I = -\sum_{p=1}^{r} \sum_{i_1 < i_2 < \cdots < i_p} \prod_{j \in \{i_1, i_2, \ldots, i_p\}} \Theta(q_j \cdot \delta) \cdot I_{i_1 i_2 \cdots i_p},$$
$$I_{i_1 i_2 \cdots i_p} = -\sum_{q=p+1}^{r} \sum_{i_{p+1} < \cdots < i_q} \prod_{j \in \{i_{p+1}, \ldots, i_q\}} \Theta(q_j \cdot \delta_{i_1 i_2 \cdots i_q}) \cdot I_{i_1 i_2 \cdots i_q},$$

which are solved by [50]

$$I_{i_1 i_2 \cdots i_p} = (-)^{r-p} \sum_{i_{p+1} < \cdots < i_r} \left[\prod_{j \in \{i_{p+1}, \ldots, i_r\}} \Theta(q_j \cdot \delta_{i_1 \cdots \hat{\jmath} \cdots i_r}) \right] \cdot I_{i_1 i_2 \cdots i_r}, \tag{C.2.3}$$

where $\hat{\jmath}$ stands for the omission of j. In particular,

$$I = (-1)^r \sum_{i_1 < i_2 < \cdots < i_r} \left[\prod_{j \in \{i_1, i_2, \ldots, i_r\}} \Theta(q_j \cdot \delta_{i_1 \cdots \hat{\jmath} \cdots i_r}) \right] \cdot I_{i_1 i_2 \cdots i_r}. \tag{C.2.4}$$

So the pole $u_{i_1} = u_{i_2} = \cdots = u_{i_r} = 0$ contributes to the JK-residue integral I if $q_j \cdot \delta_{i_1 \cdots \hat{\jmath} \cdots i_r} > 0$ for all $j \in \{i_1, i_2, \ldots, i_r\}$. We can rephrase the above result into the main rule of JK-residue prescription as follows.

- The pole $u_{i_1} = u_{i_2} = \cdots = u_{i_r} = 0$ contributes to the JK-residue integral I if η is contained in a cone spanned by the r charge vectors $q_{i_1}, q_{i_2}, \ldots, q_{i_r}$, or equivalently if

$$\eta = \sum_{k=1}^{r} c_k q_{i_k}, \qquad c_1, c_2, \ldots, c_r > 0. \tag{C.2.5}$$

One can easily confirm that this is equivalent to what is concluded after (C.2.4). From (C.2.1) and (C.2.2), we have

$$0 < \eta \cdot \delta_{i_1 \cdots \hat{\jmath} \cdots i_r} = \sum_{k=1}^{r} c_k q_{i_k} \cdot \delta_{i_1 \cdots \hat{\jmath} \cdots i_r} = c_j q_j \cdot \delta_{i_1 \cdots \hat{\jmath} \cdots i_r}, \tag{C.2.6}$$

thus $\Theta(\delta_{i_1 \cdots \hat{\jmath} \cdots i_r}) = \Theta(c_j)$.

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